1975

Comparative Studies of the Computational Analysis of One Dimensional Gas Flow

Johnny Ziebarth
Eastern Illinois University

PAPER CERTIFICATE #2

TO: Graduate Degree Candidates who have written formal theses.

SUBJECT: Permission to reproduce theses.

The University Library is receiving a number of requests from other institutions asking permission to reproduce dissertations for inclusion in their library holdings. Although no copyright laws are involved, we feel that professional courtesy demands that permission be obtained from the author before we allow theses to be copied.

Please sign one of the following statements:

Booth Library of Eastern Illinois University has my permission to lend my thesis to a reputable college or university for the purpose of copying it for inclusion in that institution's library or research holdings.

August 4, 1975
Date

I respectfully request Booth Library of Eastern Illinois University not allow my thesis be reproduced because ______________________________

____

____

______________________________ ______________________________
Date Author

pdm

Comparative Studies of the Computational Analysis

of One Dimensional Gas Flow

(TITLE)

BY

Johnny Ziebarth

THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

Master of Arts

IN THE GRADUATE SCHOOL, EASTERN ILLINOIS UNIVERSITY

CHARLESTON, ILLINOIS

1975

YEAR

I HEREBY RECOMMEND THIS THESIS BE ACCEPTED AS FULFILLING

THIS PART OF THE GRADUATE DEGREE CITED ABOVE

Aug 1st 75

DATE

August 4, 1975

DATE

ACKNOWLEDGMENT

I wish to express my sincerest appreciation to Professor S. K. Dey of the Mathematics Department of Eastern Illinois University who guided me through this thesis. The many hours of discussion we had constituted the basis for my knowledge in computational fluid dynamics. I am grateful to Professor A. J. DiPietro, Head of the Mathematics Department of Eastern Illinois University, without whose help the publication of this thesis would have been much more difficult. I wish also to express my appreciation to Professor Musfequr Rahman and Professor Roy Meyerholtz for the time they spent proofreading my work.

I wish also to express my appreciation to my wife Ms. Debra Ziebarth and her sister Ms. Amy Grigsby for their patience and skill in typing the thesis in such a short period of time.

Finally, I wish to express my gratitude again to my wife and also to my parents, Mr. and Ms. E. H. Ziebarth for their guidence and support in my academic advancement.

INTRODUCTION

In this research, a comparative analysis of various numerical techniques is done to approximate the solution of the one-dimensional, non-linear gas dynamics equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad (1)$$

subject to the initial and boundary conditions

$$u(x,0) = x \quad , \quad 0 \leq x \leq 1$$

$$u(0,t) = 0 \quad , \quad t \geq 0 \quad .$$

This non-linear equation does accept an analytical solution; therefore, the numerical solutions to be developed in this paper will be compared to this exact solution. An explicit difference scheme, an implicit difference scheme, the upwind differencing technique, the leap-frog technique, and a completely non-linear technique developed by Dey will be compared to the exact solution. Although the primary purpose of this work is to investigate the computational stability of these finite difference schemes, included in this paper is a mathematical study of the stability of the explicit and implicit techniques.

The Appendix includes both the FØRTRAN programs used in this research and profile curves at various time steps and for various values of Δt , where Δt is the magnitude of the distance between each time step.

ANALYTICAL SOLUTION

The gas dynamics equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

can be written

$$\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial u^2}{\partial x} \tag{2}$$

subject to the conditions

$$u(x,0) = x \quad , \quad 0 \leq x \leq 1$$
$$u(0,t) = 0 \quad , \quad t \geq 0 \ .$$

To solve it analytically, we seek a normal solution of the form:

$$u(x,t) = X(x) \cdot T(t)$$

which can be written

$$u = X \cdot T$$

and also

$$u^2 = X^2 \cdot T^2 \quad .$$

Now differentiating with respect to t and then x we have

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt} \quad , \tag{3}$$

and

$$\frac{\partial u^2}{\partial x} = 2X \frac{dX}{dx} \cdot T^2 \quad ; \tag{4}$$

therefore substitution of (3) and (4) in (2) yields

$$X \frac{dT}{dt} = -\frac{1}{2}\left[T^2 \cdot 2 \cdot X \frac{dX}{dx}\right]$$

or

$$\frac{1}{T^2} \frac{dT}{dt} = -\frac{dX}{dx} \quad .$$

Let

$$\frac{dX}{dx} = \lambda \tag{5}$$

so

$$\frac{dT}{T^2} = -\lambda dt \quad .$$

Integration yields,

$$-\frac{1}{T} = -\lambda t + C$$

or

$$T = \frac{1}{\lambda t - C} \quad . \tag{6}$$

From the initial condition, $u(x,0) = x$; and the boundary condition, $u(0,t) = 0$ we see that

$$X(x) \cdot T(0) = x$$

and

$$X(0) \cdot T(t) = 0 \quad .$$

Therefore,

$$X(0) = 0 \quad . \tag{7}$$

From equation (5) we have

$$dX = \lambda \, dx \quad ;$$

integration yields

$$X = \lambda x + c$$

or

$$X(x) = \lambda x + c \quad .$$

Using (7) we see that $c = 0$ and so

$$X(x) = \lambda x \quad . \tag{8}$$

Since $u(x,t) = X(x) \cdot T(t)$ we use (6) and (8) to show

$$u(x,t) = \frac{\lambda x}{\lambda t - c}$$

or

$$u(x,t) = \frac{x}{t - \frac{c}{\lambda}} \quad .$$

By the condition $u(x,0) = x$ we see that

$$-\frac{c}{\lambda} = 1$$

which yields

$$u(x,t) = \frac{x}{t+1} \quad ,$$

the exact solution of the gas dynamics equation.

In order to generate the exact solution let

$$t = t_n \quad , \quad x = x_i$$

so

$$u(x_i, t_n) = \frac{x_i}{t_n + 1}$$

Let $\Delta t = k$ and $\Delta x = h$ so

$$t_n = n \cdot \Delta t = n \cdot k$$

and

$$x_i = i \cdot \Delta x = i \cdot h \quad .$$

Introduce now a function U_i^n to represent $u(x_i, t_n)$ at $x_i = ih$ and $t_n = nk$; therefore:

$$U_i^n = \frac{ih}{1+nk} \quad ,$$

where h is given by $h = \frac{1}{I}$ and k can be chosen > 0 . Computationally then for $n = 1$ (first time step) and $i = 1, 2, \ldots, I$ we compute

$$U_1^1, U_2^1, U_3^1, \ldots, U_I^1 \quad ;$$

etc. to $n = N$ where $N > 0$.

Graphically if a particular time step, n, is plotted with one axis U_i^n and the other X_i , a straight line is generated. This is indeed what happens as indicated in Appendix Graph #1. A comparative study of this exact solution to numerical solutions is done in the Comparative Study section. Program #1 in the Appendix solves (1) for the exact solution.

NUMERICAL SOLUTIONS

In order to solve this partial differential equation numerically we will use finite difference methods to represent this equation as a difference equation. Each numerical technique will be developed and the stability of the explicit and implicit will be studied mathematically.

EXPLICIT

Consider Taylor's expansion of the following,

$$u(x, t+\Delta t) = u(x,t) + \frac{\Delta t}{1!}\left(\frac{\partial u}{\partial t}\right)_{x,t} + \frac{\Delta t^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x,\,t+\Theta\Delta t}$$

where

$$0 < \Theta < 1 \quad .$$

If we truncate the expansion after the second term we have,

$$u(x,t+\Delta t) \simeq u(x,t) + \frac{\Delta t}{1}\frac{\partial u}{\partial t} + E_T \tag{9}$$

where E_T is the truncation error or order $(\Delta t)^2$ and looks like

$$E_T \simeq \frac{\Delta t^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x,\,t+\Theta\Delta t} \quad . \tag{10}$$

We need to study under what conditions this truncation error will be small enough to neglect. Notice in (10) that if $\frac{\partial^2 u}{\partial t^2}$ is not bounded, then the truncation error is very large and our approximation in (9) is bad. Therefore we must assume the consistency condition that u is continuously differentiable at least three times with respect to x, and at least two times with respect to t. Represent this by

$$u \in C^{3,2} \quad .$$

This assumption which is indeed correct and reasonable assures us that $\frac{\partial^2 u}{\partial t^2}$ is bounded. Hence, from (10) we see that as $\Delta t \to 0$ the truncation error $\to 0$. We can now approximate from (9) that

$$\frac{\partial u}{\partial t} \simeq \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}$$

.

Again, as in the analytical solution, let $t = t_n$, $x = x_i$, $\Delta t = k$; so

$$\left(\frac{\partial u}{\partial t}\right)_i^n \simeq \frac{u(x_i, t_n + k) - u(x_i, t_n)}{k} ;$$

using U_i^n to represent $u(x_i, t_n)$ at $x_i = ih$, $t_n = nk$ we have

$$\frac{\partial u}{\partial t} \simeq \frac{U_i^{n+1} - U_i^n}{k} . \tag{11}$$

Now consider the following two Taylor expansions:

$$u(x+\Delta x, t) = u(x,t) + \frac{\Delta x}{1!}\left(\frac{\partial u}{\partial x}\right)_{x,t} + \frac{\Delta x^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_{x,t} + \tag{12}$$

$$\frac{\Delta x^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_{x+\phi\Delta x, t} \quad \text{where} \quad 0<\phi<1 ,$$

and

$$u(x-\Delta x, t) = u(x,t) - \frac{\Delta x}{1!}\left(\frac{\partial u}{\partial x}\right)_{x,t} + \frac{\Delta x^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_{x,t} - \tag{13}$$

$$\frac{\Delta x^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_{x+\lambda\Delta x, t} \quad \text{where} \quad 0<\lambda<1 .$$

Subtract (13) from (12) and get

$$u(x+\Delta x, t) - u(x-\Delta x, t) = 2\Delta x \left(\frac{\partial u}{\partial x}\right)_{x,t} + E_T$$

where

$$E_T = \frac{\Delta x^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_{x+\phi\Delta x,\, t} + \frac{\Delta x^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_{x+\lambda\Delta x,\, t}$$

and is of the order $(\Delta x)^3$.

By the assumption that $u \in C^{3,2}$ we know that $\frac{\partial^3 u}{\partial x^3}$ is bounded, therefore the truncation error $\to 0$ as $\Delta x \to 0$;

so

$$\frac{\partial u}{\partial x} \simeq \frac{u(x+\Delta x, t) - u(x-\Delta x, t)}{2\cdot\Delta x} .$$

Let $t = t_n$, $x = x_i$ and $\Delta x = h$; therefore

$$\left(\frac{\partial u}{\partial x}\right)_i^n \simeq \frac{u(x_i+h, t_n) - u(x_i-h, t_n)}{2\cdot h} .$$

Again we represent $\frac{\partial u}{\partial x}$ as

$$\frac{\partial u}{\partial x} \simeq \frac{U_{i+1}^n - U_{i-1}^n}{2h} . \qquad (14)$$

Finally from (11) and (14) we have the difference equation

$$\frac{U_i^{n+1} - U_i^n}{k} + U_i^n \frac{U_{i+1}^n - U_{i-1}^n}{2h} = 0 \qquad (15)$$

to represent

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad .$$

You can see in (15) that the time derivative is approximated by forward differences and the space derivative by central differences at (x_i, t_n) . We have approximated the continuous field over which (1) is defined by a discrete field called a grid. Now that the explicit formula has been developed, we will see how the solution is generated. Consider Diagram 1. as the grid surface to be investigated.

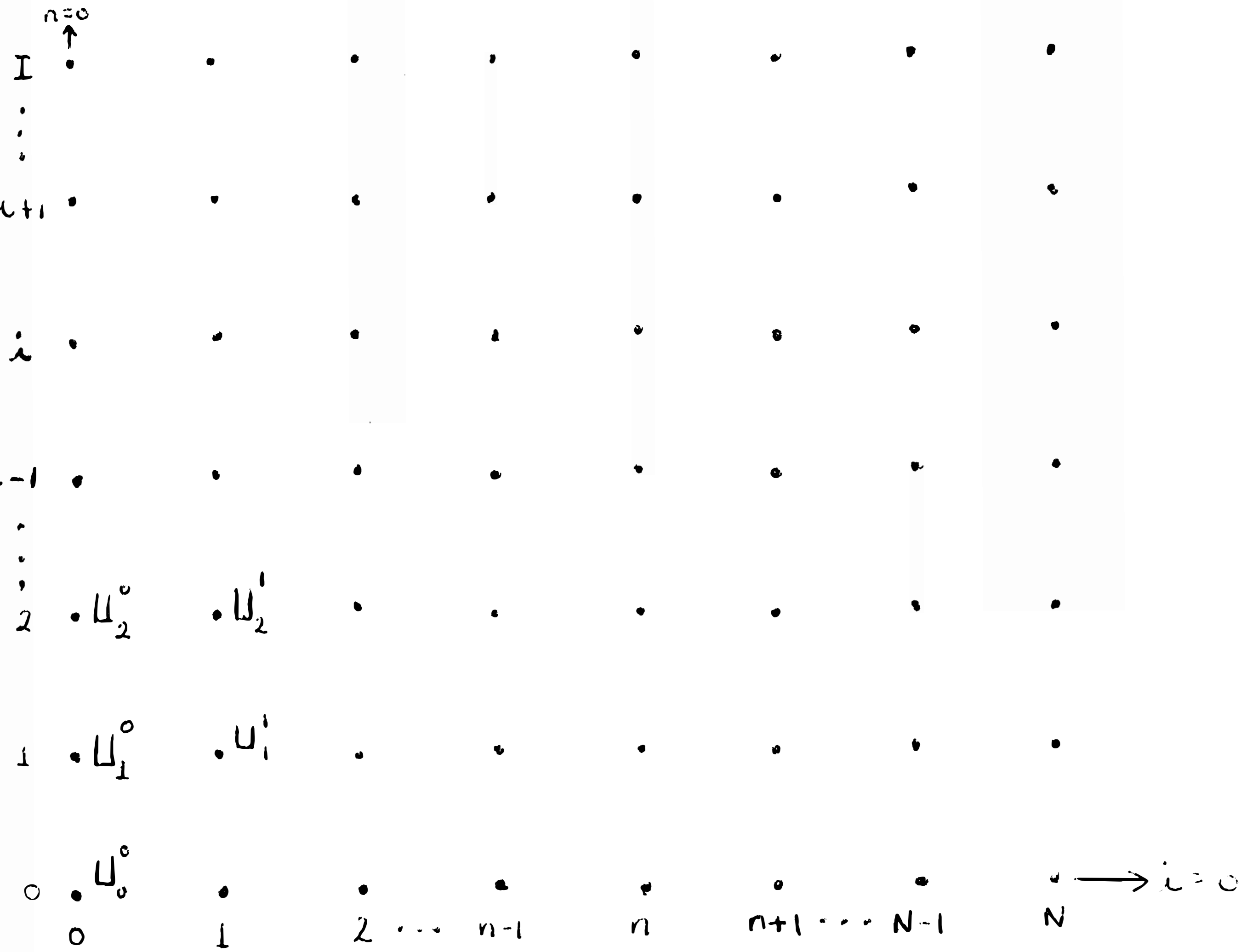


DIAGRAM 1

From the initial condition $u(x,0) = x$ and the boundary condition $u(0,t) = 0$ we can find the points indicated in red in Diagram 1 where $u(x_i, t_n) \equiv U_i^n$ subject to the conditions: $x_i = i \cdot h$, $t_n = n \cdot k$, $h = \Delta x$, and $k = \Delta t$.

Now solve (15) for U_i^{n+1}

$$U_i^{n+1} = U_i^n - \frac{k}{2h} U_i^n \left(U_{i+1}^n - U_{i-1}^n \right) \qquad (16)$$

If $n = 0$ and we let $i = 1$ on the left side of (16) we have U_1^1. From Diagram 1 we see that U_1^1 is not known. In order to calculate the value of U_1^1 we see from (16) that we need U_1^0, U_2^0, U_0^0, k, and h; all of which are known from the initial and boundary conditions. Now let $i = 2$ and calculate U_2^1; etc. to $i = I$. Then take $n = 2$ and $i = 1, 2, \ldots, I$. This algorithm will generate all the points of the grid as is shown in Program #2 of the Appendix. Note that along $i = I + 1$ for all $n = 1, 2, \ldots, N$ the values are not known for U but that to calcule U_I^n at any $n = 1, \ldots, N$ we need $U_{I+1}^1, U_{I+1}^2, \ldots, U_{I+1}^n$. This reflects the fact that the gas dynamics problem is a first order equation and hence does not require two boundary conditions. In order to generate the numerical solutions we therefore need to generate this extra boundary condition. This is done by assuming the exact solution and generating from it these points. This can be represented as

$$U_{I+1}^n = \frac{(I+1) \cdot h}{1 + nk} \qquad \text{for all } n = 1, 2, \ldots, N.$$

MATHEMATICAL STABILITY OF EXPLICIT TECHNIQUE

In the Comparative Study section, the computational stability of the explicit formula is discussed. In this section the mathematical stability of the explicit finite difference scheme will be examined.

In order to study the stability of these finite difference calculations, we must find conditions under which

$$\left| u_i^n - U_i^n \right| \leq 1$$

the difference between the theoretical and numerical solutions of the difference equation, remains bounded as n increases, Δt remaining fixed for all i . Note that for $i = 0$ or $n = 0$ their difference is zero,

$$\left| u_0^n - U_0^n \right| = 0$$

$$\left| u_i^0 - U_i^0 \right| = 0$$

therefore on the boundaries the error equals zero as would be expected.

Now we will use the von Neumann method to determine the stability of the explicit formula. Rewrite (16) as

$$U_i^{n+1} = \left(\frac{-k}{2h} U_i^n \right) U_{i+1}^n + U_i^n + \left(\frac{k}{2h} U_i^n \right) U_{i-1}^n . \tag{17}$$

Substitute

$$U_i^n = e^{\alpha n k} e^{\sqrt{-1}\beta i h} = \xi^n e^{\sqrt{-1}\beta i h}$$

where
$$\xi = e^{\alpha k}$$

and equation (17) becomes

$$\xi^{n+1} e^{\sqrt{-1}\,\beta i h} = \frac{-kU}{2h}\,\xi^{n} e^{\sqrt{-1}\,\beta(i+1)h} + \xi^{n} e^{\sqrt{-1}\,\beta i h}$$
$$+ \frac{kU}{2h}\,\xi^{n} e^{\sqrt{-1}\,\beta(i-1)h} \quad ;$$

where
$$\left(U_i^n\right)_{\substack{MAX\\ i,n}} = U$$

Cancellation of $\xi^{n} e^{\sqrt{-1}\beta h}$ yields

$$\xi = \frac{-kU}{2h} e^{\sqrt{-1}\,\beta h} + 1 + \frac{kU}{2h} e^{-\sqrt{-1}\,\beta h}$$
$$= 1 - \frac{ikU}{h}\left(\frac{e^{i\beta h} - e^{-i\beta h}}{2i}\right)$$
$$= 1 - i\,\frac{kU}{h}\sin\beta h \quad .$$

The magnitude of the amplification factor ξ is given by

$$|\xi| = \sqrt{1 + \frac{k^2U^2}{h^2}\sin^2\beta h}$$

and the requirement for stability is $|\xi| \leq 1$ for all βh . However

$$|\xi|_{MAX} = \sqrt{1 + \frac{k^2 U^2}{h^2}}$$

which is always greater than one; therefore mathematically the explicit method is not stable for the gas dynamics equation. More discussion of this computational stability is included in the <u>Comparative Study</u> section.

IMPLICIT TECHNIQUE

The implicit technique and the stability analysis which follows it are a direct result of theoretical work done by Dey.

Recall that in the explicit technique only one element at the $(n + 1)$ time step is specified in terms of some known elements at the nth time step. By an implicit technique we mean a formula in which two or more unknown elements at the $(n + 1)$ time step are specified in terms of known values at the nth or lower time step. This means that if there are say P number of unknown quantities at the $(n + 1)$ time step, then the implicit formula must be applied P times at the $(n + 1)$ time step to evaluate P unknowns. This requires the solution of P simultaneous equations to determine P number of unknowns. Obviously the computational time required to solve this set of equations is greater than that which was required by the explicit technique.

Consider the Taylor expansion which follows:

$$u(x, t-\Delta t) = u(x,t) - \frac{\Delta t}{1!}\left(\frac{\partial u}{\partial t}\right)_{x,t} + \frac{\Delta t^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x,\, t+\theta\Delta t}$$

where

$$0 < \theta < 1 \quad .$$

Again we will assume $u \in C^{3,2}$. If truncation takes place after the second term we have a truncation error, E_T , of the order $(\Delta t)^2$ which is approximately equal to

$$E_T \simeq \frac{\Delta t^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x,\, t+\theta\Delta t} \quad .$$

$u \in C^{3,2}$ implies that $\frac{\partial^2 u}{\partial t^2}$ is bounded and therefore $E_T \to 0$ as $\Delta t \to 0$.

Hence
$$\frac{\partial u}{\partial t} \simeq \frac{u(x,t) - u(x, t-\Delta t)}{\Delta t} .$$

Let $t = t_n$, $x = x_i$, $\Delta t = k$, $\Delta x = h$ and use U_i^n to represent $u(x_i, t_n)$ at $x_i = ih$, $t_n = nk$; so

$$\frac{\partial u}{\partial t} \simeq \frac{U_i^n - U_i^{n-1}}{k} .$$

Hence, in the implicit finite difference scheme the time derivative has been approximated by backward differences.

The space derivative will be approximated by central differences just as in the explicit technique. Assuming again that u is continuously differentiable at least three times with respect to x, we can truncate the Taylor expansions after the third terms and approximate

$$\frac{\partial u}{\partial x} \simeq \frac{U_{i+1}^n - U_{i-1}^n}{2h} .$$

So the gas dynamics equation when expressed in the implicit finite difference form is

$$\frac{U_i^n - U_i^{n-1}}{k} + U_i^n \frac{U_{i+1}^n - U_{i-1}^n}{2h} = 0 . \quad (18)$$

Rewriting (18) we have

$$\frac{-kU_i^n}{2h} U_{i-1}^n + U_i^n + \frac{kU_i^n}{2h} U_{i+1}^n = U_i^{n-1}$$

Let $n = 1$ and generate the equations for $i = 1, 2, 3, \ldots, I$

if $i = 1$

$$\frac{-kU_1^1}{2h} U_0^1 + U_1^1 + \frac{kU_1^1}{2h} U_2^1 + 0 \cdot U_3^1 + \cdots + 0 \cdot U_I^1 = U_2^0$$

if $i = 2$

$$\frac{-kU_2^1}{2h} U_1^1 + U_2^1 + \frac{kU_2^1}{2h} U_3^1 + 0 \cdot U_4^1 + \cdots + 0 \cdot U_I^1 = U_2^0$$

if $i = I$

$$0 \cdot U_1^1 + \cdots + 0 \cdot U_{I-2}^1 \; \frac{-kU_I^1}{2h} U_{I-1}^1 + U_I^1 + \frac{kU_I^1}{2h} U_{I+1}^1 = U$$

letting:

$$\frac{-kU_i^n}{2h} = \frac{-kY_i^n}{2h} = \alpha_i \qquad \text{where} \qquad Y_i^n = U_i^{n-1}$$

we have

$$U_1^1 + \alpha_1 U_2^1 + 0 \cdot U_3^1 + \ldots + 0 \cdot U_I^1 = U_1^0 + \alpha_2 U_0^1$$

$$-\alpha_2 U_1^1 + U_2^1 + \alpha_2 U_3^1 + 0 \cdot U_4^1 + \ldots + 0 \cdot U_I^1 = U_2^0$$

$$\vdots$$

$$0 \cdot U_1^1 + \ldots + 0 \cdot U_{I-3}^1 - \alpha_{I-1} U_{I-2}^1 + U_{I-1}^1 + \alpha_{I-1} U_I^1 = U_{I-1}^0$$

$$0 \cdot U_1^1 + \ldots + 0 \cdot U_{I-2}^1 - \alpha_I U_{I-1}^1 + U_I^1 = U_I^0 - \alpha_I U_{I+1}^1$$

In matrix form we have

$$\begin{bmatrix} 1 & \alpha_1 & 0 & \cdots & & 0 \\ -\alpha_2 & 1 & \alpha_2 & 0 & \cdots & 0 \\ & \cdot & \cdot & \cdot & & \\ 0 & \cdots & 0 & -\alpha_{I-1} & 1 & \alpha_{I-1} \\ 0 & \cdots & & 0 & -\alpha_I & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_I \end{bmatrix}^1 = \begin{bmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_I \end{bmatrix}^0 + \begin{bmatrix} \alpha_1 U_0^1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\alpha_I U_{I+1}^1 \end{bmatrix}$$

or

$$\underline{W} \cdot \underline{U}^n = \underline{U}^{n-1} + \underline{C}$$

where $\underline{U}^{n-1}$ and $\underline{C}$ are known.

Since $\underline{W} = \underline{W}(\underline{U}^n)$, the matrix equations are clearly nonlinear. To solve the system, $\underline{U}^n$ will be replaced by an approximation and $\underline{W}(\underline{U}^n)$ will be linearized.

LINEARIZED EXISTENCE & STABILITY

In order to show that

$$\underline{U}^n = \begin{bmatrix} U_1^n \\ U_2^n \\ \cdot \\ \cdot \\ \cdot \\ U_I^n \end{bmatrix}$$

has a solution in

$$\underline{W} \cdot \underline{U}^n = \underline{U}^{n-1} + \underline{C} \tag{19}$$

it is necessary to prove $\underline{W}$ is nonsingular so that

$$\underline{U}^n = \underline{W}^{-1} (\underline{U}^{n-1} + \underline{C}) .$$

Given that $\underline{W}$ is the tri-diagonal matrix in (19) we must prove the following theorem.

THEOREM 1: Prove that $\underline{W}$ is nonsingular and its eigenvectors are linearly independent.

PROOF: We know $|\det \underline{W}| = \lambda_1 \lambda_2 \ldots \lambda_I$ where λ_i equals an eigenvalue of $\underline{W}$; therefore to show that $\underline{W}$ may be nonsingular it must be true that

$$|\det \underline{W}| \neq 0$$

or, in other words, $\lambda_i \neq 0$ for all i = 1, 2, . . . , I .

Thus we need to find the eigenvalues of $\underline{W}$. If these eigenvalues are distinct, the corresponding eigenvectors will be linearly independent and will form a basis for the vector space of dimension I, V^I , where $U^n \in V^I$ for all n .

Represent the vector equation as

$$\begin{bmatrix} 1 & r & 0 & 0 & 0 & \cdots & 0 \\ -r & 1 & r & 0 & 0 & \cdots & 0 \\ 0 & -r & 1 & r & 0 & \cdots & 0 \\ & \cdot & \cdot & \cdot & & & \\ 0 & \cdots & 0 & -r & 1 & r & \\ 0 & \cdots & 0 & 0 & -r & 1 & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_I \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_I \end{bmatrix} \tag{20}$$

A general term from this system of difference equations is

$$-rx_{j-1} + x_j + rx_{j+1} = \lambda x_j \tag{21}$$

Note that this is a second order difference equation and thus requires two conditions to solve it. From (20) it is reasonable and true that these two conditions can be

$$x_0 = 0$$

$$x_{I+1} = 0$$

Let $x_j = \beta^j$

so from (21)

$$-r\beta^{j-1} + \beta^j + r\beta^{j+1} - \lambda\beta^j = 0$$

$$-r\frac{1}{\beta} + (1-\lambda) + r\beta = 0$$

$$\beta^2 + \frac{1-\lambda}{r}\beta - 1 = 0$$

and by the quadratic formula

$$\beta = \frac{-\left(\frac{1-\lambda}{r}\right) \pm \sqrt{\left(\frac{1-\lambda}{r}\right)^2 + 4}}{2}$$

$$= \frac{\lambda - 1}{2r} \pm \sqrt{\left(\frac{\lambda-1}{2r}\right)^2 + 1} \quad . \qquad (22)$$

Let
$$\frac{\lambda-1}{2r} = i \cos \phi$$

therefore
$$\left(\frac{\lambda-1}{2r}\right)^2 = -\cos^2 \phi$$

and
$$1 + \left(\frac{\lambda-1}{2r}\right)^2 = 1 - \cos^2\phi = \sin^2 \phi \quad .$$

So from (22)

$$\begin{aligned} \beta &= i\cos\phi \pm \sin\phi \\ &= i(\cos\phi \mp i\sin\phi) \\ &= i\,e^{\mp i\phi} \end{aligned}$$

and we have

$$\beta_1 = i\,e^{i\phi}$$

$$\beta_2 = i\,e^{-i\phi} \quad ,$$

thus the eigenvectors will be

$$X_k = A\beta_1^k, + B\beta_2^k$$

$$X_k = i^k\ (Ae^{ik\phi} + Be^{-ik\phi}). \tag{23}$$

If $k = 0$ and $x_0 = 0$ then from (23) we have

$$0 = A + B$$

which implies $A = -B$;

and if $k = I + 1$, $x_k = 0$ we have

$$0 = i^{I+1}\ A\ (e^{i(I+1)\phi} - e^{-i(I+1)\phi})$$

$$0 = i^{I+1}\ A \cdot 2i \cdot \sin\ (I+1)\ \phi\ .$$

Since $\sin\ (I+1)\ \phi = 0 = \sin m\pi,$

$$(I+1)\ \phi = m\pi$$

so $\phi = \dfrac{m\pi}{I+1}$

therefore

$$\lambda_m = 1 + i2r \cdot \cos \frac{m\pi}{I+1} \quad , \tag{24}$$

Let $(I + 1)\ h = a$

so $\lambda_m = 1 + 2 \cdot i \cdot r \cos \dfrac{m\pi h}{a}$

this shows that $\lambda_m \neq 0$ for any m .

Hence all λ_m's are distinct and non-zero; this implies $\underline{W}$ is nonsingular. Q.E.D.

We have thus shown that the solution to (19) does exist. In order to continue and to show the stability of this implicit technique, we need the following theorem:

<u>Theorem 2:</u> If $\underline{W}$ is nonsingular, then the eigenvalues of $\underline{W}^{-1}$ are equal to $\frac{1}{\lambda_m}$, where λ_m are the eigenvalues of $\underline{W}$.

<u>Proof:</u>

Given: $\underline{W}\cdot\underline{X} = \lambda\,\underline{X}$

therefore $\underline{W}^{-1}\cdot\underline{W}\,\underline{X} = \lambda\,\underline{W}^{-1}\cdot\underline{X}$

and $\underline{X} = \lambda\,\underline{W}^{-1}\cdot\underline{X}$

so $\frac{1}{\lambda}\underline{X} = \underline{W}^{-1}\cdot\underline{X}.$ <u>Q.E.D.</u>

Now from (24)

$$\lambda_m^{-1} = \left(1 + i2r\cdot\cos\frac{m\pi h}{a}\right)^{-1}$$

$$= \frac{1}{1 + i2r\cdot\cos\frac{m\pi h}{a}}.$$

By multiplying the numerator and denominator by

$$1 - i2r\cdot\cos\frac{m\pi h}{a}$$

we have

$$\frac{1}{\lambda_m} = \frac{1 - i2r\cdot\cos\frac{m\pi h}{a}}{1 + 4r^2\cdot\cos^2\frac{m\pi h}{a}}. \tag{25}$$

Let

$$\frac{1}{\lambda_m} = \zeta_m.$$

We have that $\underline{U}$ and $\underline{X} \in V^I$ where $\underline{X}$ spans V^I

therefore

$$\underline{U}^{(1)} = \sum \alpha_m \chi_m$$

$$\underline{W}^{-1}\ \underline{U}^{(1)} = \sum \alpha_m \underline{W}^{-1} \chi_m$$

$$= \sum \alpha_m \xi_m \chi_m$$

also

$$\underline{W}^{-1}\ \underline{U}^{(2)} = \sum \alpha_m \xi_m \underline{W}^{-1} \chi_m$$

$$= \sum \alpha_m \xi_m^2 \chi_m$$

and so on all the way to

$$\underline{W}^{-1}\ \underline{U}^{(n)} = \sum \alpha_m \xi_m^n \chi_m .$$

If $|\xi_m| \leq 1$ the error will not be amplified, If $|\xi_m| > 1$ the error will be amplified.

Since

$$|\xi_m| = \frac{a - ib}{a^2 + b^2} = \frac{\sqrt{a^2+b^2}}{a^2+b^2} = \frac{1}{\sqrt{a^2+b^2}}$$

we see from (25) that

$$|\xi_m| = \frac{1}{\sqrt{1 + 4r^2 \cos^2 \frac{m\pi h}{a}}} \leq 1 \quad \text{for all } k, h.$$

This shows that the implicit finite difference scheme will be stable for the gas dynamics equation for all values of Δt.

Now that the implicit scheme has been developed, its existence shown, and its stability proven, we will investigate how to solve this tridiagonal system of equations. Recall the system of equations which needs to be solved is

$$
\begin{aligned}
&b U_1^n + c U_2^n + 0 \cdot U_3^n + \dots + 0 \cdot U_I^n = U_1^{n-1} - a U_0^n \\
&a U_1^n + b U_2^n + c U_3^n + 0 U_4^n + \dots + 0 U_I^n = U_2^{n-1} \\
&\text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\
&0 \cdot U_1^n + \dots 0 \cdot U_{I-3}^n + a U_{I-2}^n + b U_{I-1}^n + c U_I^n = U_{I-1}^{n-1} \\
&0 U_1^n + \dots 0 U_{I-2}^n + a U_{I-1}^n + b U_I^n = U_I^{n-1} - c U_{I+1}^n
\end{aligned}
\tag{26}
$$

where

$$a = \frac{-U_i^n k}{2h}$$

$$b = 1$$

$$c = \frac{U_i^n k}{2h}$$

In order to solve for all the values of U_i^n, $n = 1, 2, \dots, N$; $i = 1, 2, \dots, I$ in (26) we begin by setting $n = 1$ and solving for U_i^1, $i = 1, 2, \dots, I$ then we go to $n = 2$ and solve for U_i^2, $i = 1, 2, \dots, I$ etc., all the way to $n = N$ and solve for U_i^N, $i = 1, 2, \dots, I$. Notice in (26) that all values on the right side of the equality signs are known either from the initial and boundary conditions or from values already calculated.

the values

$$a = \frac{-U_i^n \cdot k}{2h} \qquad \text{and} \qquad c = \frac{U_i^n \cdot k}{2h}$$

do however present a problem since they contain a value of the velocity which is not yet known. We therefore make the assumption that

$$a = \frac{-Y_i^n k}{2h} \qquad \text{and} \qquad c = \frac{Y_i^n k}{2h}$$

where

$$Y_i^n = U_i^{n-1} .$$

With this assumption all the values of a and c can be computed and the system of equations (26) can be solved.

After the system of equations has been solved for U_i^n $i = 1, 2, \ldots, I$, these new values of U_i^n are put in place of the assumed values Y_i^n. New values of a and c are computed and the system of equations is recalculated. This process is continued until the condition

$$\left| \left(U_i^n\right)_{k+1} - \left(U_i^n\right)_K \right|_{MAX} < L$$

is satisfied. This condition requires that the largest differences between a new value of $\left(U_i^n\right)_{k+1}$ and a previously used value $\left(U_i^n\right)_K$ for $i = 1, 2, \ldots,$ I be less than an arbritrarily picked value L . In program #3 $L = .001$. After the values of U_i^1 , $i = 1, 2, \ldots, I$ are computed in this manner, n is set equal to 2 and the process is repeated. This continues for all values of n .

Another interesting problem is the solution of the tridiagonal system. When the system of equations (26) are put in matrix form, they look like this:

$$\begin{bmatrix} b_1 & -c_1 & & & \\ -a_2 & b_2 & -c_2 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -a_{I-1} & b_{I-1} & -c_{I-1} \\ & & & -a_I & b_I \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{I-1} \\ U_I \end{bmatrix} = \begin{bmatrix} \delta_1 + a_1 U_0 \\ \delta_2 \\ \vdots \\ \delta_{I-1} \\ \delta_I + c_I U_{I+1} \end{bmatrix},$$

where $$a = \frac{U_i^n k}{2h}, \quad b = 1, \quad c = \frac{-U_i^n k}{2h}, \quad \delta_i = U_i^{n-1}$$

and U_0^n and U_{I+1}^n are known from the boundary conditions. Necessary conditions are that $a_i > 0$, $b_i > 0$ and $b_i \geq (a_i + c_i)$, for $1 \leq i \leq I$ and $c_i > 0$.

The method for solving this system is as follows:

Consider the difference relation

$$U_i^n = V_i U_{i+1}^n + P_i \qquad \text{for } 0 \leq i \leq I$$

it follows that

$$U_{i-1}^{n} = V_{i-1} U_i^{n} + P_{i-1} .$$

This can be used to eliminate U_{i-1} from the difference formula

$$-a_i U_{i-1}^{n} + b_i U_i^{n} - c_i U_{i+1}^{n} = \delta_i$$

which defines the tri-diagonal system. As a result

$$U_i^{n} = \frac{c_i}{b_i - a_i V_{i-1}} U_{i+1}^{n} + \frac{\delta_i + a_i P_{i-1}}{b_i - a_i V_{i-1}}$$

is obtained; hence

$$V_i = \frac{c_i}{b_i - a_i V_{i-1}} , \quad P_i = \frac{\delta_i + a_i P_{i-1}}{b_i - a_i V_{i-1}} . \tag{27}$$

Since $U_0^n = 0$ this makes $V_0 = P_0 = 0$.

All other values of V_i, $i = 1, 2, \ldots, I$ and P_i, $i = 1, 2, \ldots, I$ can now be calculated from (27) .

Since U_{I+1}^n is known, the values of $U_1^n, U_2^n, \ldots, U_I^n$ can be calculated from

$$U_I^n = V_I U_{I+1}^n + P_I$$

$$U_{I-1}^n = V_{I-1} U_I^n + P_{I-1}$$

$$\vdots$$

$$U_1^n = V_1 U_2^n + P_1 .$$

In order that large errors do not appear in the calculated values of U_i^n for $i = 1, 2, \ldots, I$ the condition

$$\left| V_i \right| \leq 1, \qquad i = 1, 2, \ldots, I$$

must hold. But since $a_i > 0$, $b_i > 0$, $c_i > 0$ and $b_i \geq (a_i + c_i)$ for $1 \leq i \leq 1$ the relation

$$0 < V_i \leq 1 \quad ; \quad i = 1, 2, \ldots, I$$

holds.

Although this method is equivalent to Gaussian elimination, it reduces some of the error generated by the back substitution in the Gaussian method. This method also minimizes the amount of storage required by the computer. Program #3, found in the appendix, was used for these computations.

More discussion of the computational results is done in the section Comparative Study.

DEY'S TECHNIQUE

The following numerical method for solving the gas dynamics equation was developed by Dey. The theoretical analysis of the method along with the stability analysis are yet to be done. Before this non-linear method is applied to the gas dynamics problem, a simplified explanation of the technique will be discussed.

Suppose we have

$$f(x_1, x_2) = b_1$$
$$g(x_1, x_2) = b_2 \quad .$$

These equations may be expressed as:

$$x_1 = F(x_1, x_2) \quad \text{where} \quad \left|\frac{\partial F}{\partial x_1}\right| < 1$$

and

$$x_2 = G(x_1, x_2) \quad \text{where} \quad \left|\frac{\partial G}{\partial x_2}\right| < 1$$

assume some

$$x_1^{(0)} \quad , \quad x_2^{(0)}$$

so

$$x_1^{(1)} = \gamma_1^{(1)} + F\left(x_1^{(0)}, x_2^{(0)}\right) \tag{28}$$

and

$$x_2^{(1)} = \gamma_2^{(1)} + G\left(x_1^{(1)}, x_2^{(0)}\right)$$

where the superscript (0) implies the initial assumed value and (1) implies the first calculated value; γ_1 and γ_2 are not yet known.

Now if we let $x_1 = \xi_1$, $x_2 = \xi_2$ be the solutions then

$$\xi_1 = F(\xi_1, \xi_2)$$

and $$\xi_2 = G(\xi_1, \xi_2) .$$

If we assume $$x_1^{(1)} + \xi_1 \quad \text{and} \quad x_2^{(1)} = \xi_2$$

then $$F(x_1^{(1)}, x_2^{(1)}) = F(\gamma_1 + F_{0,0}, x_2^{(1)})$$

where $$F_{0,0} = F(x_1^{(0)}, x_2^{(0)})$$

and $$x_1^{(1)} \simeq F(\gamma_1 + F_{0,0}, x_2^{(0)}) .$$

Now by Taylor's expansion and truncating after the second term

$$\gamma_1^{(1)} + F_{0,0} = F(F_{0,0}, x_2^{(0)}) + \gamma_1^{(1)} \left(\frac{\partial F}{\partial x_1}\right)_{F_{0,0},\, x_2^{(0)}}$$

or

$$\gamma_1^{(1)} = \frac{F(F_{0,0}, x_2^{(0)}) - F_{0,0}}{1 - \left(\frac{\partial F}{\partial x_1}\right)_{F_{0,0},\, x_2^{(0)}}}$$

and similarly

$$\gamma_1^{(n)} = \frac{F(F_{n-1,n-1}, x_2^{(n-1)}) - F_{n-1,n-1}}{1 - \left(\frac{\partial F}{\partial x_1}\right)_{F_{n-1,n-1},\, x_2^{n-1}}}$$

where

$$F_{n,n} = F\,(x_1^{\,n},\ x_2^{\,n})\,.$$

Now γ_2 can be computed as follows.

$$G\,(x_1^{(1)},\ x_2^{(1)}) = G\,(x_1^{(1)},\ \gamma_2 + G_{1,0})$$

where

$$G_{1,0} = G\,(x_1^{(1)},\ x_2^{(0)})$$

and

$$x_2^{(1)} = G\,(x_1^{(1)},\ \gamma_2 + G_{1,0})\,.$$

By Taylor's expansion and truncating after the second term

$$\gamma_2^{(1)} + G_{1,0} = G\,(x_1^{(1)}, G_{1,0}) + \gamma_2^{(1)} \left(\frac{\partial G}{\partial x_2}\right)_{x_1^{(1)},\, G_{1,0}}$$

or

$$\gamma_2^{(1)} = \frac{G\,(x_1^{(1)}, G_{1,0}) - G_{1,0}}{1 - \left(\frac{\partial G}{\partial x_2}\right)_{x_1^{(1)},\, G_{1,0}}}$$

and similarly

$$\gamma_2^{(n)} = \frac{G\,(x_1^{(n)}, G_{n,n-1}) - G_{n,n-1}}{1 - \left(\frac{\partial G}{\partial x_2}\right)_{x_1^{(n)},\, G_{n,n-1}}}$$

Using the values of $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$, $x_1^{(1)}$ and $x_2^{(1)}$ are calculated from (28). now $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$ are tested to make sure they are both less than a chosen test value. If they are not then $x_1^{(1)}$ and $x_2^{(1)}$ replace $x_1^{(0)}$ and $x_2^{(0)}$ respectively and the algorithm is begun again with equations (28).

Program #4 in the Appendix solves the gas dynamics equation by this method. The computational stability of this technique is discussed in the Comparative Study section..

UPWIND DIFFERENCING TECHNIQUE

Another method for numerically solving equation (1) is called the upwind differencing technique and will now be developed. The computational stability of this method will be discussed in the section Comparative Study.

We will begin with the gas dynamics equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

subject to $u(x,0) = x \quad , \quad 0 \leqslant x \leqslant 1$

and $u(0,t) = 0 \quad , \quad t \geqslant 0 \, .$

Applying Taylor's expansion we obtain the following

$$u(x+\Delta x, t) = u(x,t) + \frac{\Delta x}{1!}\left(\frac{\partial u}{\partial x}\right)_{x,t} + \frac{\Delta x^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_{x+\Theta\Delta x,\, t}$$

where $0 < \Theta < 1$.

Assuming again that $u \in C^{3,2}$, U is continuously differentiable at least two times with respect to t, we can truncate after the second term with the understanding that the truncation error $\rightarrow 0$ as $\Delta x \rightarrow 0$. Therfore we can approximate

$$\frac{\partial u}{\partial x} \simeq \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x}$$

or using the normal notation

$$\frac{\partial u}{\partial x} \simeq \frac{U_{i+1}^{n} - U_{i}^{n}}{h} \, .$$

Now apply the Taylor expansion to obtain

$$u(x, t+\Delta t) = u(x,t) + \frac{\Delta t}{1!}\left(\frac{\partial u}{\partial t}\right)_{x,t} + \frac{\Delta t^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x,\, t+\phi\Delta t}$$

where $0 < \phi < 1$.

Again using $u \in C^{3,2}$ so that after truncation we can approximate

$$\frac{\partial u}{\partial t} \simeq \frac{U_i^{n+1} - U_i^n}{k} .$$

From (29) and (30) we have as the finite difference approximation of (1)

$$\frac{U_i^{n+1} - U_i^n}{k} + U_i^n \frac{U_{i+1}^n - U_i^n}{h} = 0$$

or

$$U_i^{n+1} = U_i^n \left[1 - \frac{k}{h}\left(U_{i+1}^n - U_i^n \right) \right] .$$

Given the initial and boundary conditions

$$U_i^0 = ih , \qquad U_0^t = 0$$

and with the aid of a high speed computer facility we can calculate

U_i^n for $n = 1, 2, \ldots, N$; $i = 1, 2, \ldots, I$.

Program number 5 in the Appendix was used to compute these values of U.

A slight variation to this method was also attempted. Begin now with the gas dynamics equation written in the form

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0 \quad .$$

$\dfrac{\partial u}{\partial t}$ is still approximated by

$$\frac{\partial u}{\partial t} \simeq \frac{U_i^{n+1} - U_i^n}{k}$$

but $\dfrac{\partial u^2}{\partial x}$ is, by the same type of a development, approximated by

$$\frac{\partial u^2}{\partial x} = \frac{(U_{i+1}^n)^2 - (U_i^n)^2}{h}$$

The finite difference approximation becomes

$$\frac{U_i^{n+1} - U_i^n}{k} + \frac{(U_{i+1}^n)^2 - (U_i^n)^2}{2h} = 0$$

or

$$U_i^{n+1} = U_i^n - \frac{k}{2h}\left[(U_{i+1}^n)^2 - (U_i^n)^2\right]$$

subject to the same initial and boundary conditions as before. The method can be used just as easily as the previous method and program number 5 computes the solution with only the value of c changed and statement #60 changed.

LEAP-FROG TECHNIQUE

The final numerical technique which will be attempted to solve the gas dynamics equation is called the "leap-frog" method. Its computational stability will be discussed in the section Comparative Study, although it will be developed here.

Begin with

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

subject to $u(x,0) = x \quad , \quad 0 \leq x \leq 1$

and $u(0,t) = 0 \quad , \quad t \geq 0 \quad .$

Change its form to

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = 0 \quad . \tag{28}$$

Apply Taylor's expansion to obtain the following;

$$u^2(x+\Delta x, t) = u^2(x,t) + \frac{\Delta x}{1!}\left(\frac{\partial u^2}{\partial x}\right)_{x,t} + \frac{\Delta x^2}{2!}\left(\frac{\partial^2 u^2}{\partial x^2}\right)_{x,t} + \frac{\Delta x^3}{3!}\left(\frac{\partial^3 u^2}{\partial x^2}\right)_{x+\theta\Delta x, t} \tag{29}$$

where $0 < \theta < 1$

and

$$u^2(x-\Delta x, t) = u^2(x,t) - \frac{\Delta x}{1!}\left(\frac{\partial u^2}{\partial x}\right)_{x,t} + \frac{\Delta x^2}{2!}\left(\frac{\partial^2 u^2}{\partial x^2}\right)_{x,t} - \frac{\Delta x^3}{3!}\left(\frac{\partial^3 u^2}{\partial x^3}\right)_{x-\phi\Delta x, t} \tag{30}$$

where $0 < \phi < 1$.

Subtract (30) from (29). Assuming $U \in C^{3,2}$ truncation is possible and by appropriate substitutions we get

$$\frac{\partial u^2}{\partial x} \simeq \frac{\left(U_{i+1}^n\right)^2 - \left(U_{i-1}^n\right)^2}{2h} \qquad (31)$$

Apply Taylor's expansion to obtain

$$u(x,t+\Delta t) = u(x,t) + \frac{\Delta t}{1!}\left(\frac{\partial u}{\partial t}\right)_{x,t} + \frac{\Delta t^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x,t+\lambda\Delta t}$$

where $0 < \lambda < 1$.

Assume $U \in C^{3,2}$, then by truncation after the second term and appropriate substitutions

$$\frac{\partial u}{\partial t} \simeq \frac{U_i^{n+1} - U_i^n}{k} \qquad (32)$$

From (30), (31), and (28)

we have

$$\frac{U_i^{n+1} - U_i^n}{k} + \frac{\left(U_{i+1}^n\right)^2 - \left(U_{i-1}^n\right)^2}{4h} = 0$$

or

$$U_i^{n+1} = U_i^n - \frac{k}{4h}\left[\left(U_{i+1}^n\right)^2 - \left(U_{i-1}^n\right)^2\right]$$

subject to

$$U_i^0 = i \cdot h \qquad , U_0^t = 0 .$$

This can easily be solved for U_i^n , $i = 1, 2, \ldots, I$; $n = 1, 2, \ldots, N$ with the aid of a high speed computing facility. Program number 6 given in the Appendix was used to compute these values of U_i^n .

COMPARATIVE STUDY

This section will analyze the various numerical techniques as compared to the analytical solution of the gas dynamics equation. Computer results were calculated for all six numerical methods described and each was compared to the analytical solution. The parameter changed in the finite difference formulas was Δt. Graphs 2 through 21 in the Appendix will help in visualizing how these results compared for each Δt. Graphs were drawn for time steps 20, 40, 60, 80, and 100 for each Δt where $\Delta t = .0025$, $.005$, $.01$, and $.1$. Table One on the following page will also help as a reference to this section.

From the form of the analytical solution

$$u(x,t) = \frac{x}{1+t}$$

we can expect that its graphs are straight lines. This is indeed what happens at each time step for each of the four values of Δt.

Theoretically the explicit technique should be unstable. Recall that the stability criteria is that $|\xi| \leq 1$ where

$$|\xi|_{MAX} = \sqrt{1 + \frac{k^2 U^2}{h^2}} \qquad \text{where} \quad U = \left(U_i^n\right)_{\substack{MAX \\ i,n}} .$$

Since $k > 0$ and $U \neq 0$, mathematically this is never less than or equal to zero. Note, however, that as k gets smaller this quantity comes closer to being equal to zero, especially when U_i^n has not reached its maximum. This

	Δt = .0025	Δt = .005	Δt = .01	Δt = .1
EXACT	Graph # 2-6	Graph # 7-11	Graph # 12-16	Graph # 17-21
EXPLICIT	Graph # 2-6	Graph # 7-11	Graph # 12-16 Unstable for large i .	Unstable aborts immediately.
IMPLICIT	Graph # 2-6	Graph # 7-11	Graph # 12-16	Graph # 17-21
DEY	Graph # 2-6	Graph # 7-11	Graph # 12-16	Unstable will not run.
UPWIND DIFFER-ENCING	Unstable aborts after time step 33.	Unstable aborts after time step 18.	Unstable aborts after time step 11.	Unstable aborts after time step 3.
LEAP FROG	Graph # 2-6	Graph # 7-11	Unstable aborts immediately.	Unstable aborts immediately.

TABLE 1

is why numerically the explicit technique gave fair results for Δt = .005 and Δt = .0025 . When Δt = .1 the explicit is so unstable that results are not obtained on the computer. It can also be seen from graphs, numbers 2 through 13, that the explicit method gives best results when i is small, which is equivalent to U_i^n not at its maximum, or when Δt is small.

Both methods of solving (1) by the upwind differencing technique failed for all values of Δt . It was also seen that as Δt gets larger, the program aborts sooner. Table 1 shows that when Δt = .0025 the program gave fair results for small values of i up to the time step 45, but when $\Delta t = .1$ the program only computed results for three time steps before the program aborted.

The leap frog technique gave better results than the upwind differencing technique for $\Delta t = .0025$ and $\Delta t = .005$. However, for $\Delta t = .01$ and $\Delta t = .1$ the program aborts immediately. The leap frog technique for $\Delta t = .0025$ and $\Delta t = .005$ are included in the graphs in the appendix.

Dey's technique for solving equation (1) gave results for $\Delta t = .0025$ and $\Delta t = .005$. The results for $\Delta t = .01$ show substantial vibration around the exact solution, however they were better than the explicit solution. When $\Delta t = .1$ the program was too unstable to compute and print values of the velocity of the gas.

The implicit technique which is mathematically stable is also computationally stable for all values of Δt. When Δt does get large, however, there occurs a small vibration around the exact solution. This apparent error is very slight and does not subtract from the excellent numerical results obtained by the implicit finite difference formula.

CONCLUSION

The one-dimension non-linear gas dynamics equation does not on an initial inspection appear to be difficult to solve. It is interesting, however, to see all the normal explicit methods fail, but the implicit succeed. This gives some indication of the power of an implicit finite difference technique. The non-linear technique by Dey, although it failed for this problem, gave an indiciation of being a valid numerical technique. The vibrating motion that was produced by this method leaves open the possibility that if a damping factor can be applied to the formula, better results can easily be obtained. The vibration can clearly be seen in the graphs where $\Delta t = .01$. Note especially Graph # 16 where Dey's technique constantly vibrates while the explicit technique jumps erratically.

APPENDIX

```
C     PROGRAM 1
C     EXACT SOLUTION OF THE GAS DYNAMICS EQUATION
      REAL K
      DIMENSION U(102,102)
      READ (5,20) K,H
   20 FORMAT (2F6.4)
      WRITE (6,30) K,H
   30 FORMAT ('1','K = ',F6.4,5X,'H = ',F6.4,///)
C     THE VALUES OF U FROM I = 0 to 100 ARE NOW CALCULATED
      DO 50 N = 1,101
      DO 40 I = 1,101
   40 U(I,N) = (I -1) * H/(1.0 + (N - 1) * K)
   50 CONTINUE
C     THE VALUES OF U(I), I = 1, 2, . . ., I ARE NOW PRINTED AT EACH TIME STEP
      DO 90 NN = 1,101
      LN = NN - 1
      WRITE (6,60) LN
   60 FORMAT ('0',///,'TIME STEP =',I4,30X,'EXACT SOLUTION')
      WRITE (6,70)
   70 FORMAT ('0', 'VALUES OF U(I) ARE')
      WRITE (6,80) (U(II,NN),II = 1,101)
   80 FORMAT ('0', 10F10.5)
   90 CONTINUE
      STOP
      END
```

```
C     PROGRAM 2
C     EXPLICIT SOLUTION OF GAS DYNAMICS EQUATION
      REAL K
      DIMENSION U(103,103)
      READ (5,20) K,H,I,N
   20 FORMAT (2F10.5,2I5)
      WRITE (6,30) K,H
   30 FORMAT ('1', 'K = ', F10.5,5X,'H = ',F10.5,///)
      IJ = I - 1
      JI = I + 1
C     SET INITIAL AND BOUNDARY CONDITIONS
      DO 40 II = 1,JI
   40 U(II,1) = (II - 1)*H
      DO 50 NN = 2,N
      U(I,NN) = ((I - 1)*H)/(1. + (NN - 1)*K)
   50 U(1,NN) = 0.0
      DO 70 NN = 2,N
      DO 60 II = 2,IJ
   60 U(II,NN) = U(II,NN - 1)*(1. + K/(2.*H)*(U(II - 1,NN - 1) - U(II + 1,NN - 1)))
   70 CONTINUE
      DO 110 KN = 1,N
      LN = KN - 1
      WRITE (6,80) LN
   80 FORMAT ('0','TIME STEP = ',I4,30X,'EXPLICIT SOLUTION')
      WRITE (6,90)
   90 FORMAT ('0','VALUES OF U(I) ARE')
      WRITE (6,100) (U(IK,KN),IK = 1,I)
  100 FORMAT ('0',10F10.5)
  110 CONTINUE
      STOP
      END
```

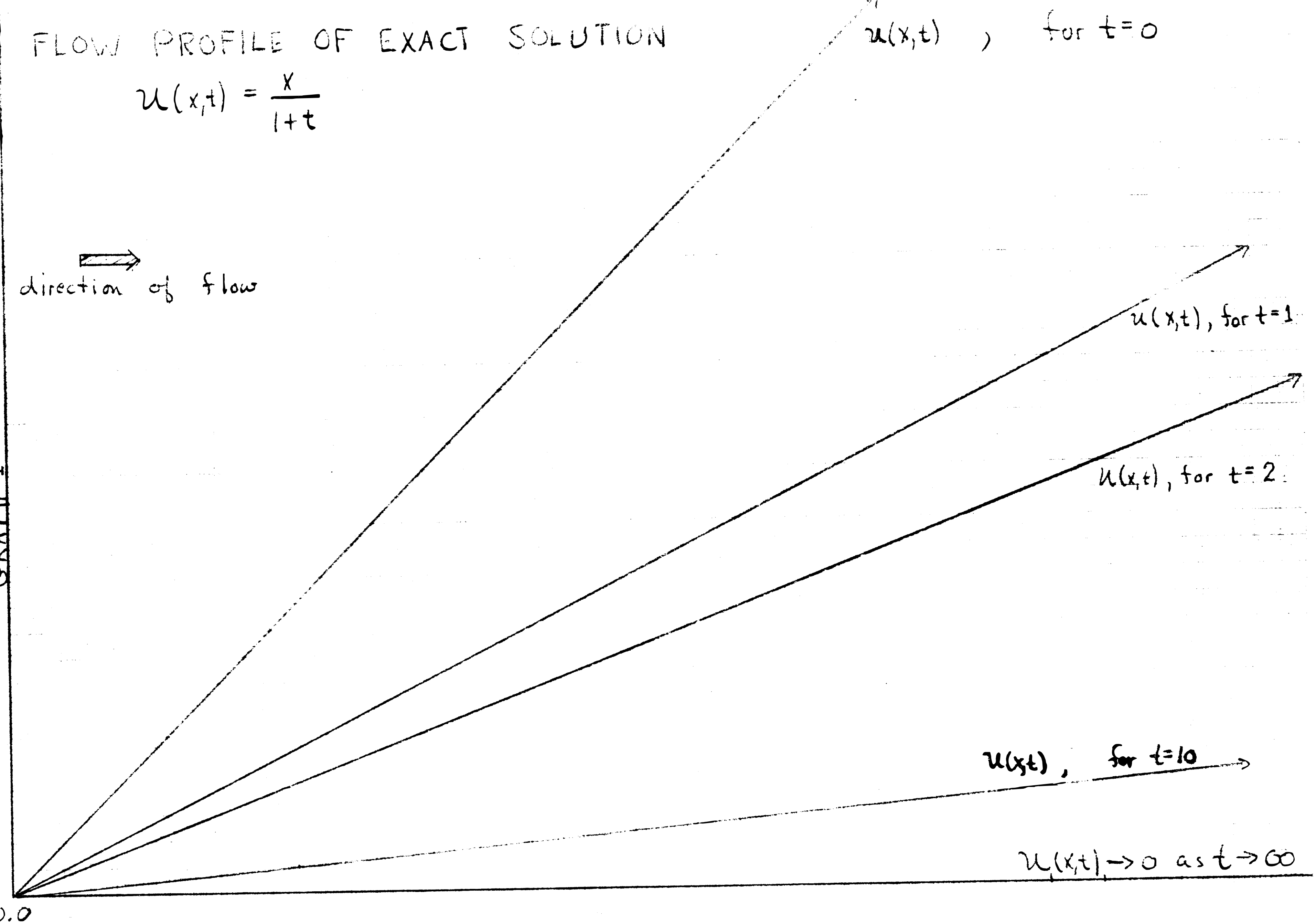
```
C      PROGRAM 3
C      IMPLICIT SOLUTION OF GAS DYNAMICS EQUATION
       REAL K
       DIMENSION U(102,102),Y(102,102),DELTA(102),V(102),P(102),A(102),C(102)
       READ (5,20) K,H,I,N
    20 FORMAT (2F10.5,2I5)
       WRITE (6,20) K,H
    30 FORMAT ('1','K = ',F10.5,5X,'H = ',F10.5,///)
       IJ = I - 1
C      SET INITIAL AND BOUNDARY CONDITIONS
       DO 40 II = 1,I
    40 U(II,1) = (II - 1)*H
       DO 50 NN = 2,N
       U (1,NN) = 0.0
    50 U(I,NN) = ((I - 1)*H)/(1. + (NN - 1)*K)
C      B IS ELEMENT IN MAIN DIAGONAL OF MATRIX
       B = 1.
C      KN IS THE TIME STEP COUNTER
       KN = 1
       IF(KN.EQ.1) GO TO 150
C      ASSUMPTIONS ARE MADE HERE SO THAT THE OTHER ELEMENTS OF THE
C      TRIDIAGONAL MATRIX CAN BE COMPUTED
    60 DO 70 II = 2,IJ
    70 Y(II,KN) = U(II,KN - 1)
C      ALGORITHM TO SOLVE TRIDIAGONAL MATRIX EQUATION IS NOW FOLLOWED
    80 DO 90 II = 2,IJ
       A(II) = Y(II,KN)*K/(2.*H)
    90 C(II) = -Y(II,KN)*K/(2.*H)
       DELTA(2) = U(2,KN - 1) + A(2)*U(1,KN)
       DO 100 II = 3,IJ
   100 DELTA (II) = U(II,KN - 1)
       DELTA (I) = U(I,KN - 1) + C(IJ)*U(I,KN)
       V(1) = 0.0
       P(1) = 0.0
       DO 110 IP = 2,IJ
   110 V(IP) = C(IP)/(B - A(IP)*V(IP - 1))
       DO 120 IP = 2,IJ
   120 P(IP) = (DELTA(IP) + A(IP)*P(IP - 1))/(B - A(IP)*V(IP - 1))
       DO 130 L = 1,IJ
       IC = I - L
   130 U(IC,KN) = V(IC)*U(IC + 1,KN) + P(IC)
       DO 140 II = 2,IJ
       AAB = ABS(U(II,KN) - Y(II,KN))
       IF (AAB.GE..001) GO TO 190
   140 CONTINUE
   150 LN = KN - 1
       WRITE (6,160) LN
   160 FORMAT ('0','TIME STEP =',I4,30X,'IMPLICIT SOLUTION")
       WRITE (6,170)
   170 FORMAT ('0','VALUES OF U(I) ARE')
       WRITE (6,180) (U(II,KN),II = 1,I)
   180 FORMAT ('0',10F10.5)
       IF (KN.EQ.N) GO TO 210
       KN = KN + 1
```

```
      GO TO 60
  190 DO 200 II = 2,IJ
  200 Y(II,KN) = U(II,KN)
      GO TO 80
  210 STOP
      END
```

```
C       PROGRAM 4
C       GAS DYNAMICS PROBLEM SOLVED BY DEYS TECHNIQUE
        INTEGER TS
        REAL K
        DIMENSION U(103,103),G(103)
C       F(N) AND P(N) ARE FUNCTIONS, P(N) IS THE PARTIAL DERIVATIVE FUNCT.
        F(N,KN) = C*U(N,KN)*(U(N - 1)KN) - U(N + 1,KN)) + U(N,KN - 1)
        P(N,KN) = C*(U(N - 1,KN) - U(N + 1,KN))
        READ (5,20) H,K,I,TS
     20 FORMAT (2F6.4,2I5)
        WRITE (6,30) H,K,I,TS
     30 FORMAT ('1',5X,'H = ',F6.4,5X,'K = ',F6.4,5X,'I = ',I5,5X,'TIME STE
       1P = ',I5,///)
C       C IS A CONSTANT IN THE PROGRAM
        C = K/(2.*H)
        III = I - 1
C       INITIAL AND BOUNDARY CONDITIONS ARE NOW SET
        DO 40 L = 1,I
     40 U(L,1) = (L - 1)*H
        DO 50 L = 2,TS
        U(1,LL) = 0.0
     50 U(I,LL) = (I - 1)*H/(1. + (LL - 1)*K)
C       TIME STEPS ARE VALUES OF KN
        KN = 1
        GO TO 120
C       ASSUME U(2,KN),U(3,KN),***,U(I,KN)
     60 DO 70 LK = 2,III
     70 U(LK,KN) = U(LK,KN - 1)
C       N REFERS TO STEPS FROM 1 to I
     80 N = 2
C       THE FOLLOWING ROUTINE COMPUTES NEW VALUES OF U
     90 V = F(N,KN)
        U(N,KN) = V
        W = F(N,KN)
        Y = W - V
        Z = P(N,KN)
C       G(N) REFERS TO GAMMA(N)
        G(N) = Y/(1. - Z)
        U(N,KN) = G(N) + V
        IF (N.EQ.III) GO TO 100
        N = N + 1
        GO TO 90
    100 DO 110 LM = 2,III
    110 IF (G(LM) .GT. 0.1E - 10) GO TO 80
    120 KNK = KN - 1
        WRITE (6,130) KNK
    130 FORMAT('0',///,' ','TIME STEP = ',I4,30X,'DEYS TECHNIQUE',///)
        WRITE (6,140)
    140 FORMAT('0','VALUES OF U',///)
        WRITE (6,150) (U(J,KN),J = 1,I)
    150 FORMAT ('0',10F10.5)
        IF(KN.EQ.101) GO TO 160
        KN = KN + 1
        GO TO 60
    160 STOP
        END
```

```
C     PROGRAM 5
C     UPWIND DIFFERENCING FOR GAS DYNAMICS EQUATION
      REAL K
      DIMENSION U(103,103)
      READ (5,20) K,H
   20 FORMAT (2F10.5)
      WRITE (6,30) K,H
   30 FORMAT ('1','K = ',F10.5,5X,'H = ',F10.5)
C     INITIAL AND BOUNDARY CONDITIONS ARE NOW SET
      DO 40 I = 1,101
   40 U(I,1) = (I - 1)*H
      DO 50 N = 2,101
      U(1,N) = 0.0
   50 U(101,N) = 100.*H/(1.*(N - 1)*K)
C     C IS CONSTANT
      C = K/H
C     CALCULATIONS ARE NOW MADE FOR VALUES OF U
      DO 100 N = 1,101
      DO 60 I = 2,100
   60 U(I,N + 1) = U(I,N) - C*(U(I + 1,N)*U(I,N) - U(I,N)**2)
      LN = N - 1
      WRITE (6,70) LN
   70 FORMAT ('0','TIME STEP = ',I3,30X,'UPWIND DIFFERENCING')
      WRITE (6,80)
   80 FORMAT('0','VALUES OF U(I) ARE')
      WRITE (6,90) (U(I,N),I = 1,101)
   90 FORMAT ('0',10F10.5)
  100 CONTINUE
      STOP
      END
```

```
C     PROGRAM 6
C     LEAP FROG SOLUTION OF GAS DYNAMICS EQUATION
      REAL K
      DIMENSION U(103,103)
      READ (5,20) K,H,I,N
   20 FORMAT (2F10.5,2I5)
      WRITE (6,30) K,H
   30 FORMAT ('1','K = ',F10.5,5X,'H = ',F10.5,///)
      IJ = I - 1
      JI = I + 1
      C + K/(4.*H)
C     SET INITIAL AND BOUNDARY CONDITIONS
      DO 40 II = 1,JI
   40 U(II,1) = (II - 1)*H
      DO 50 NN = 2,N
      U(I,NN) = ((I - 1)*H)/(1. + (NN - 1)*K)
   50 U(1,NN) = 0.0
      DO 70 NN = 2,N
      DO 60 II = 2,IJ
   60 U(II,NN) = U(II,NN - 1) -C*(U(II + 1,NN - 1)**2 - U(II - 1,NN - 1)**2)
   70 CONTINUE
      DO 110 KN = 1,N
      LN = KN - 1
      WRITE (6,80) LN
   80 FORMAT('0','TIME STEP = ',I4,30X,'LEAP FROG')
      WRITE (6,90)
   90 FORMAT ('0','VALUES OF U(I) ARE')
      WRITE (6,100) (U(IK,KN),IK = 1,I)
  100 FORMAT ('0',10F10.5)
  110 CONTINUE
      STOP
      END
```



FLOW PROFILE OF EXACT SOLUTION
$u(x,t) = \frac{x}{1+t}$
direction of flow
$u(x,t)$, for $t=0$
$u(x,t)$, for $t=1$
$u(x,t)$, for $t=2$
$u(x,t)$, for $t=10$
$u(x,t) \to 0$ as $t \to \infty$
0.0

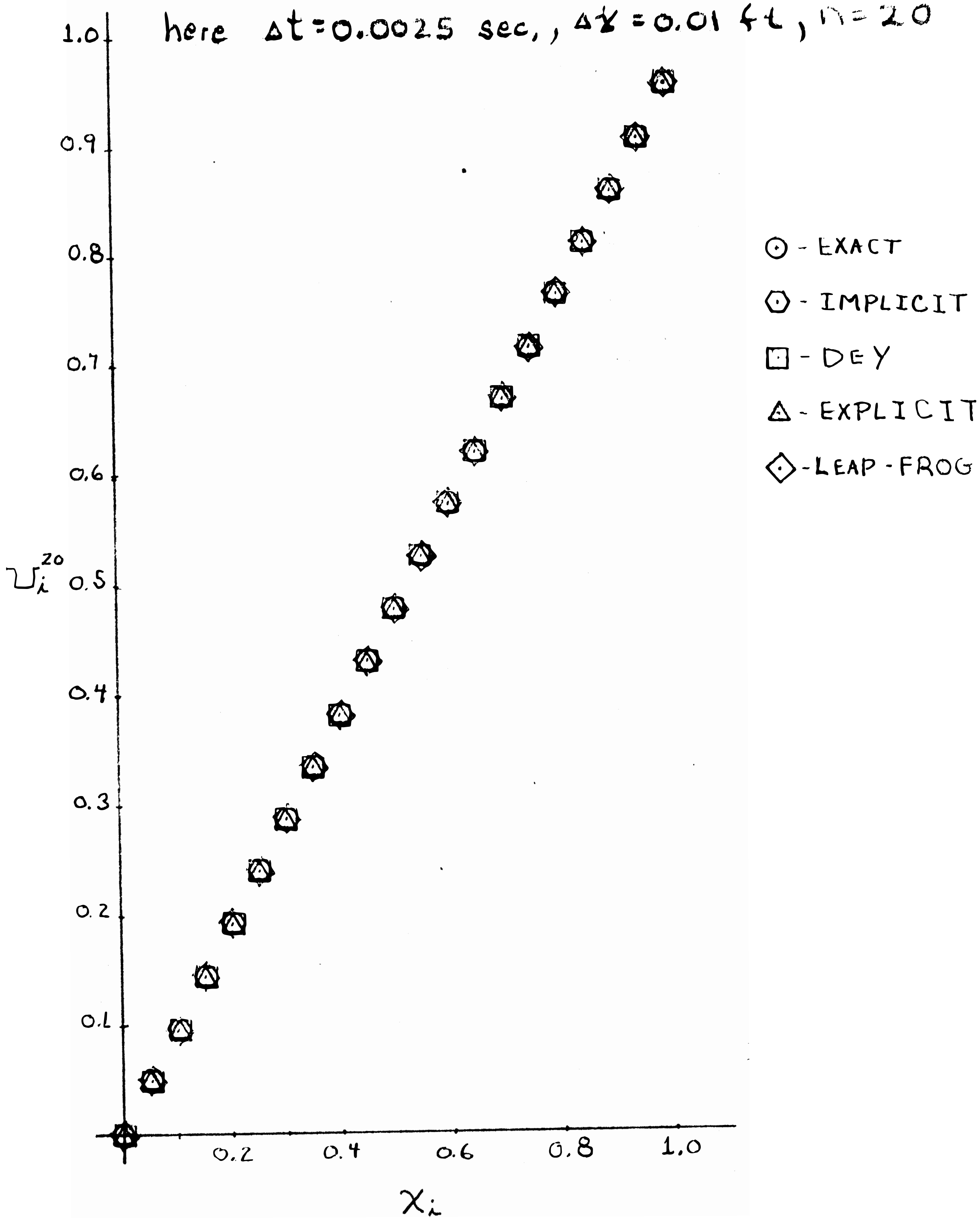
GRAPH 1

GRAPH 2

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

here $\Delta t = 0.0025$ sec., $\Delta x = 0.01$ ft, $n = 20$

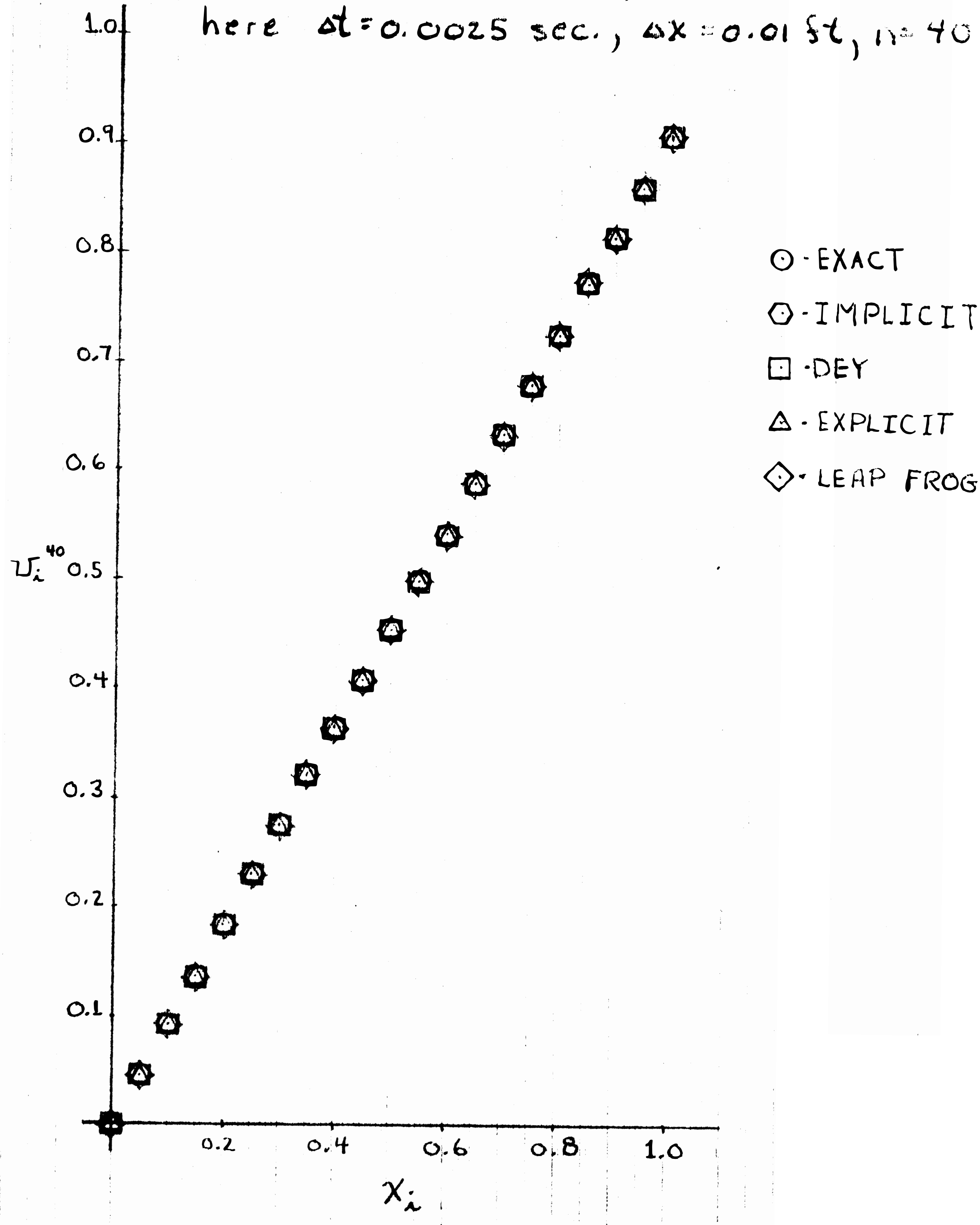


GRAPH 3

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

here $\Delta t = 0.0025$ sec., $\Delta x = 0.01$ ft, $n = 40$

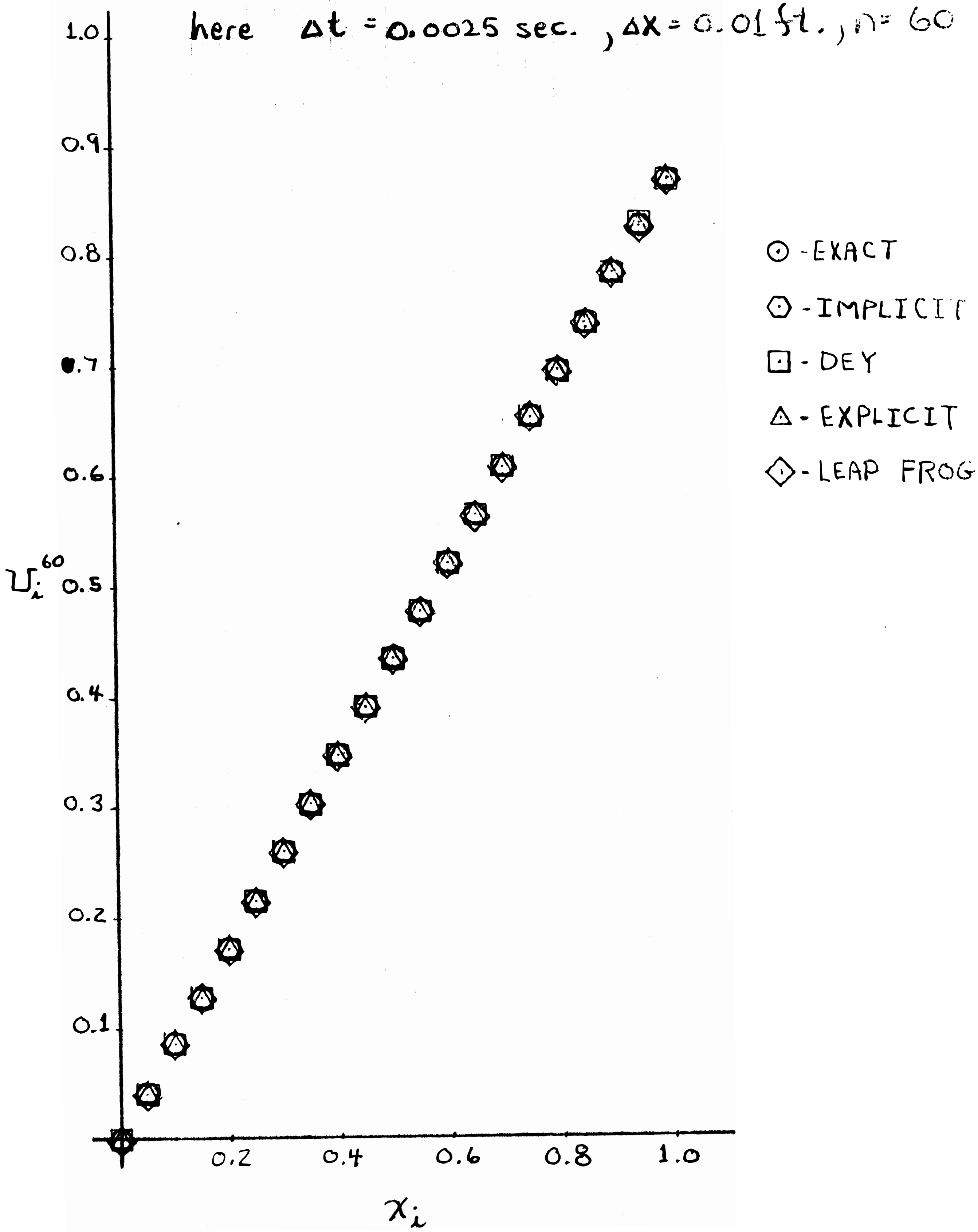


GRAPH 4

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

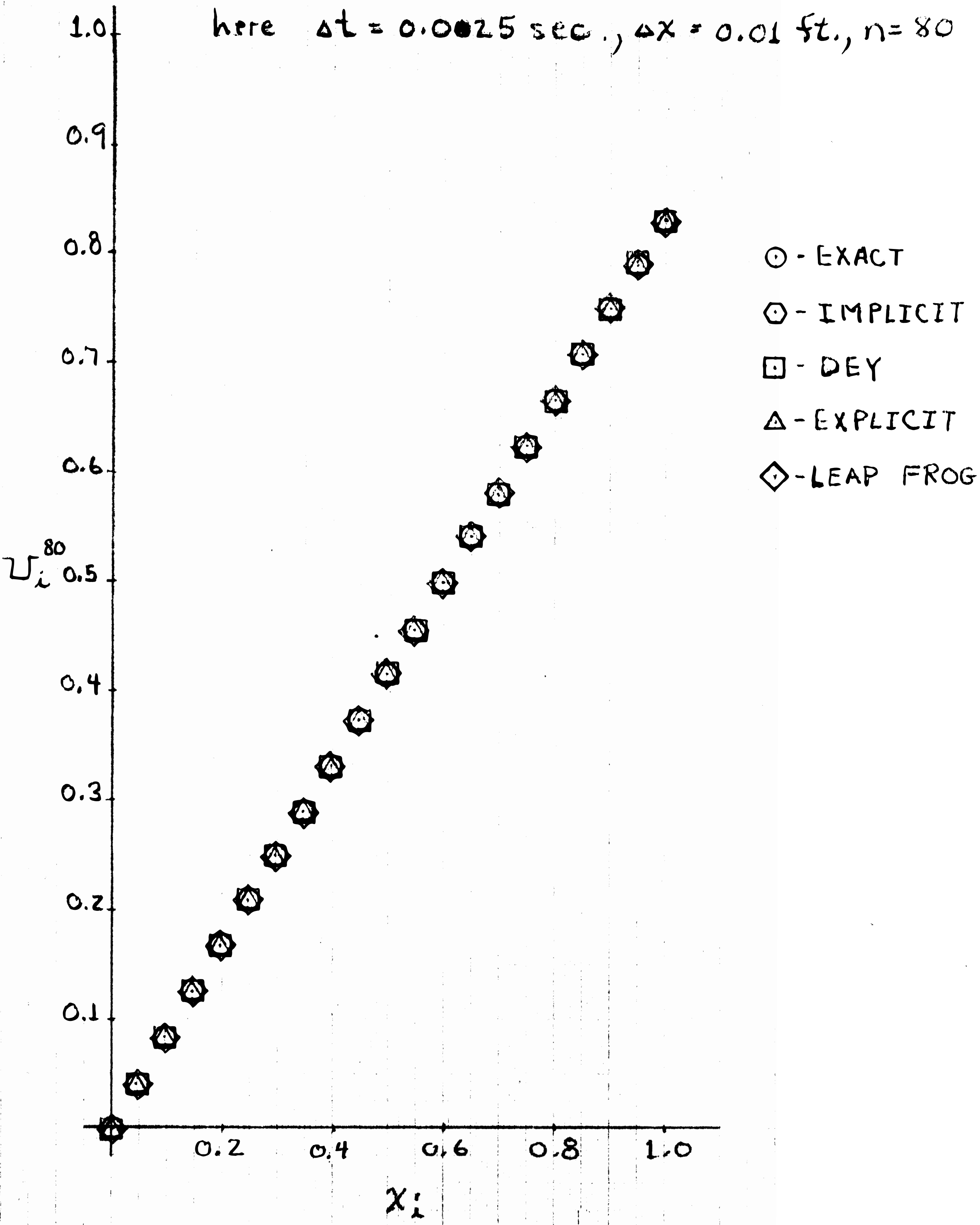
here $\Delta t = 0.0025$ sec. , $\Delta x = 0.01$ ft. , $n = 60$



GRAPH 5

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

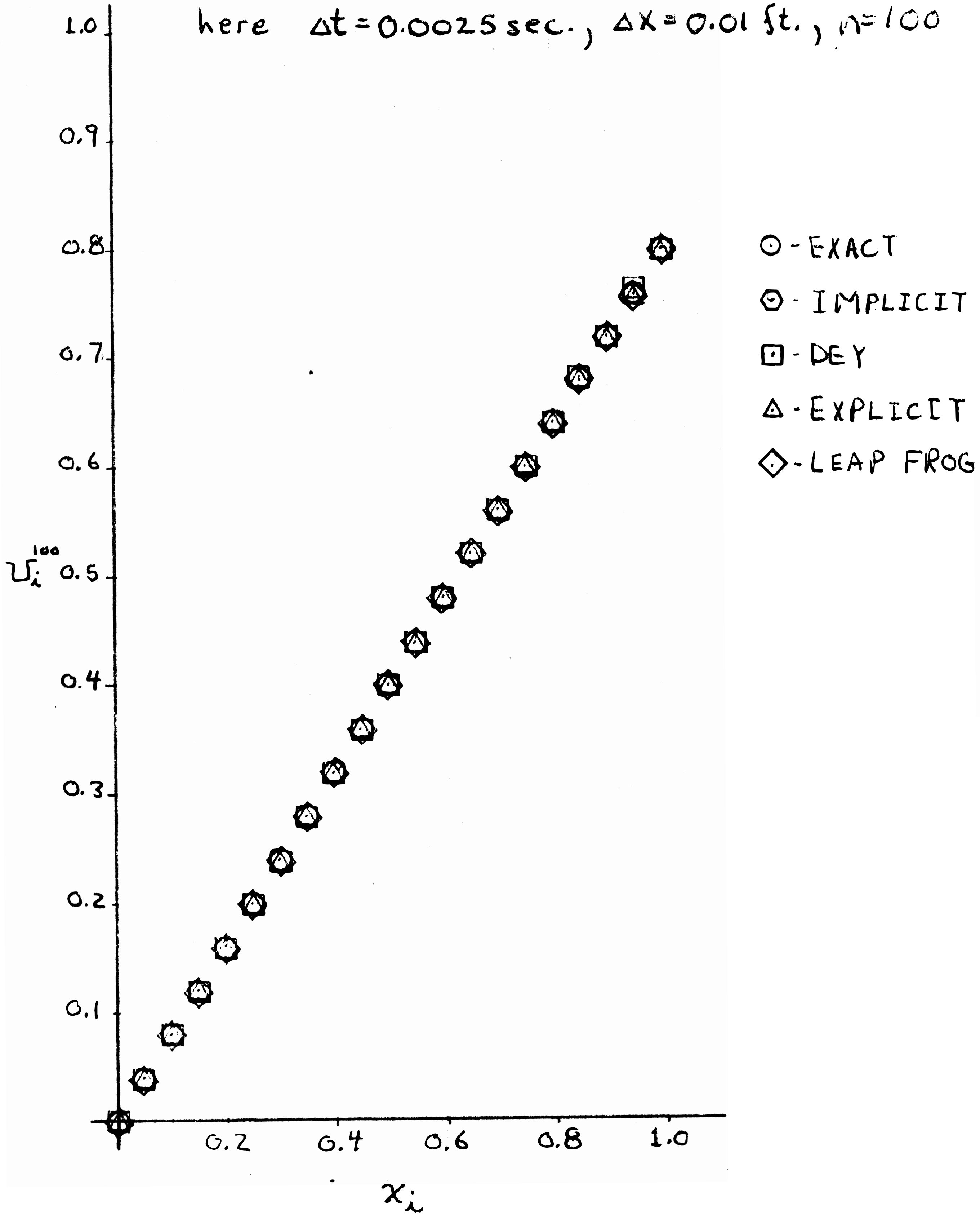


GRAPH 6

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

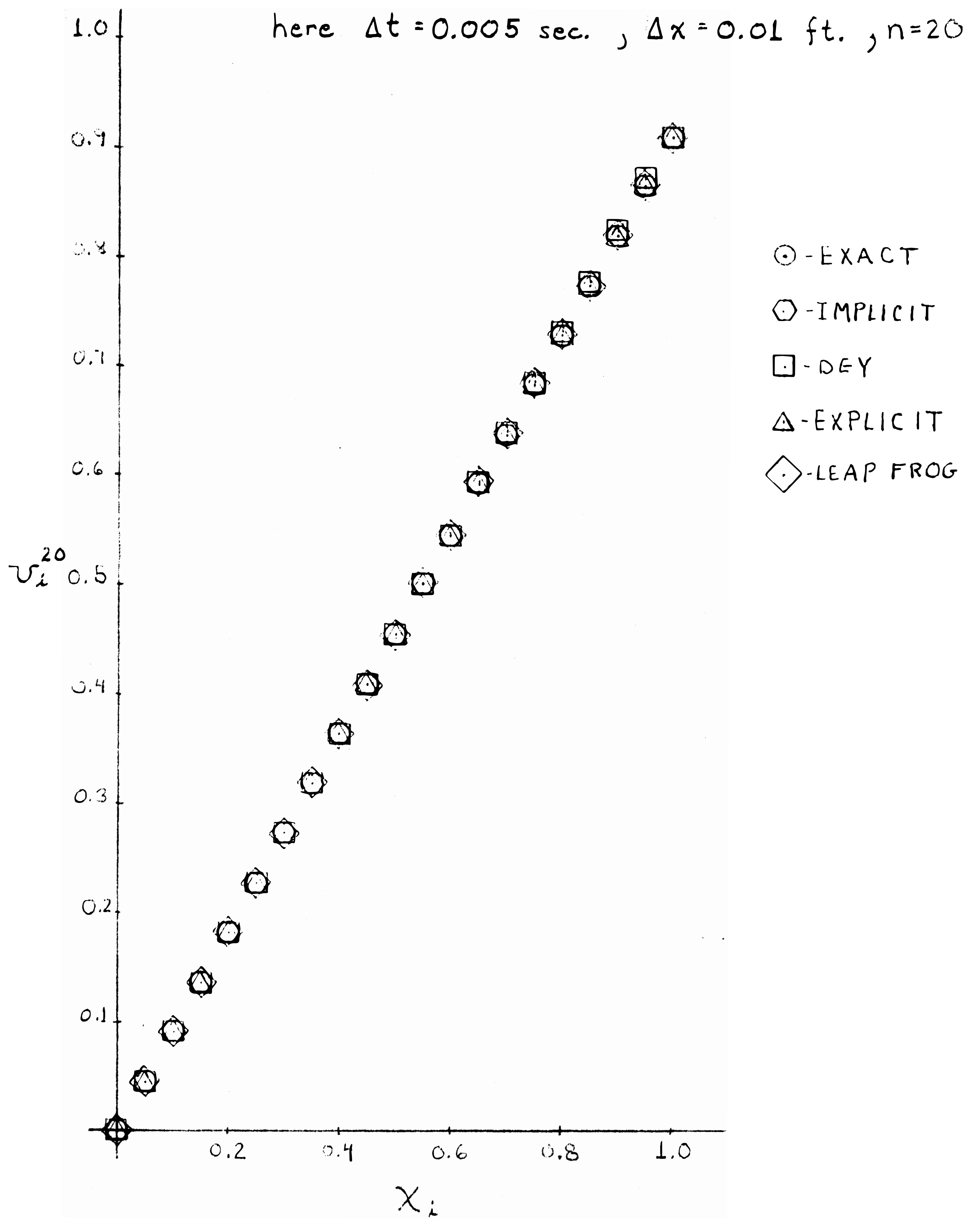
here $\Delta t = 0.0025$ sec., $\Delta x = 0.01$ ft., $n = 100$



GRAPH 7

SOLUTION OF THE GAS DYNAMICS EQUATION

$$U_i^n = U(x_i, t_n) \quad , t_n = n \cdot \Delta t \quad , x_i = i \cdot \Delta x$$

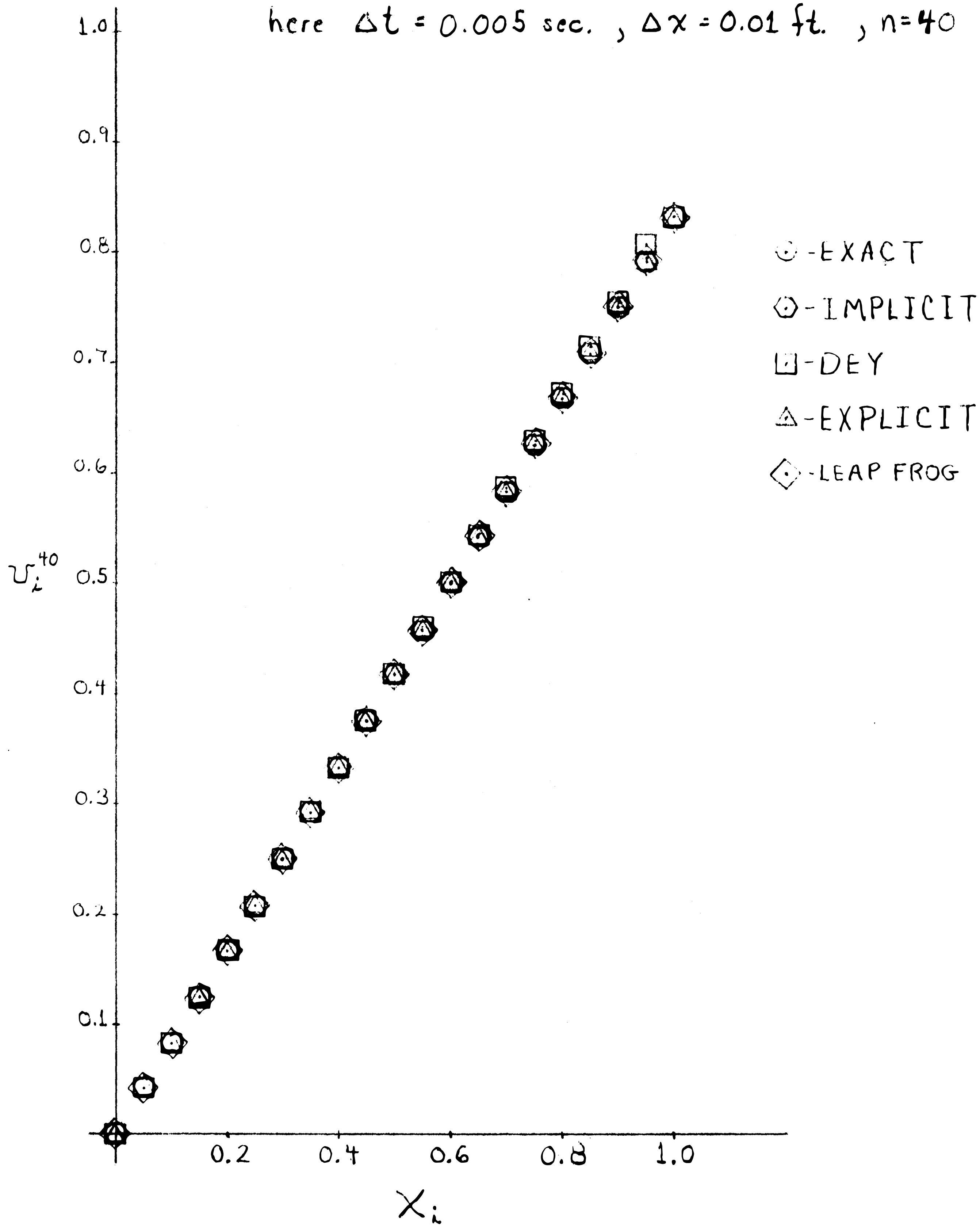


GRAPH 8

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

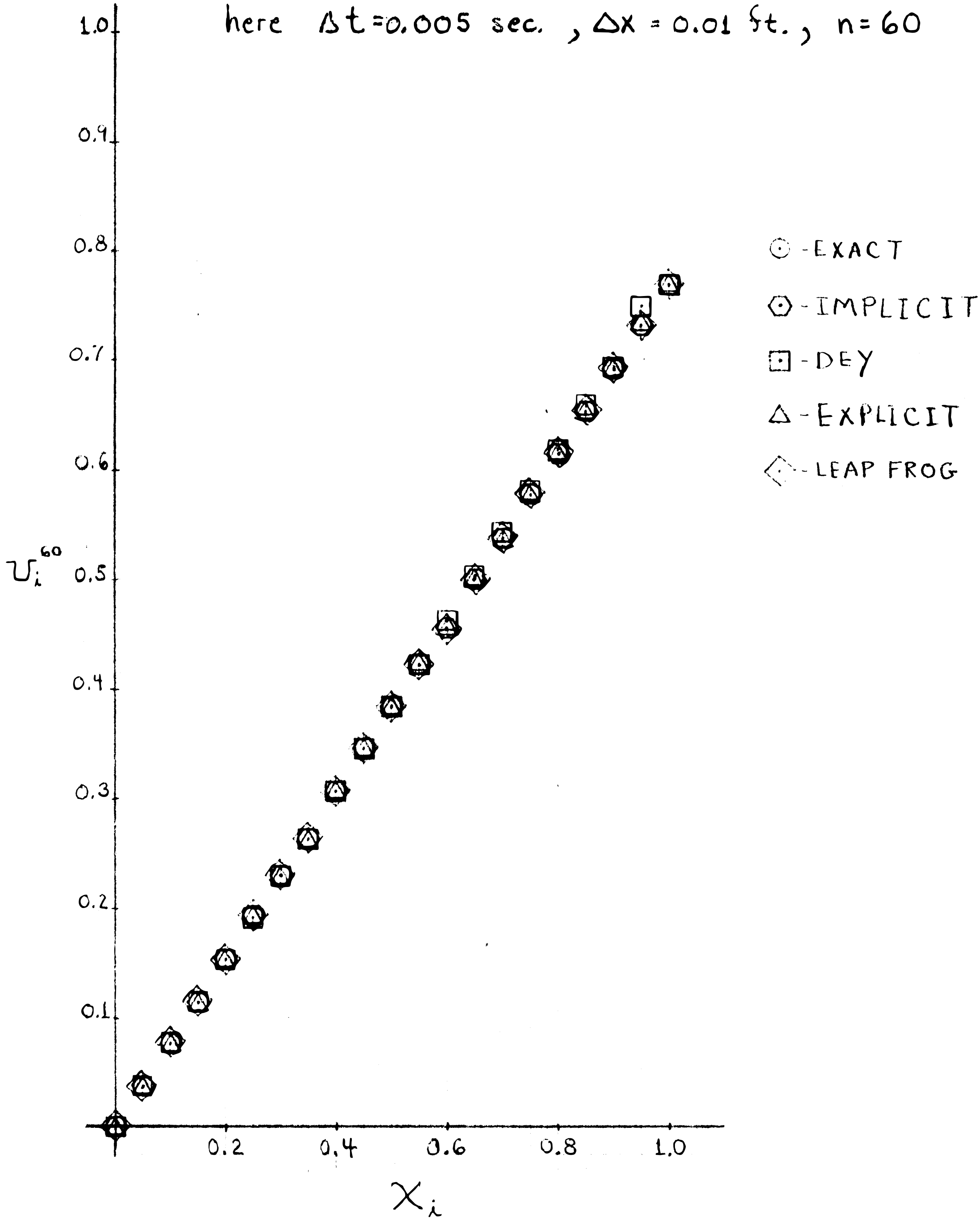
here $\Delta t = 0.005$ sec. , $\Delta x = 0.01$ ft. , $n = 40$



GRAPH 9

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $X_i = i \cdot \Delta x$

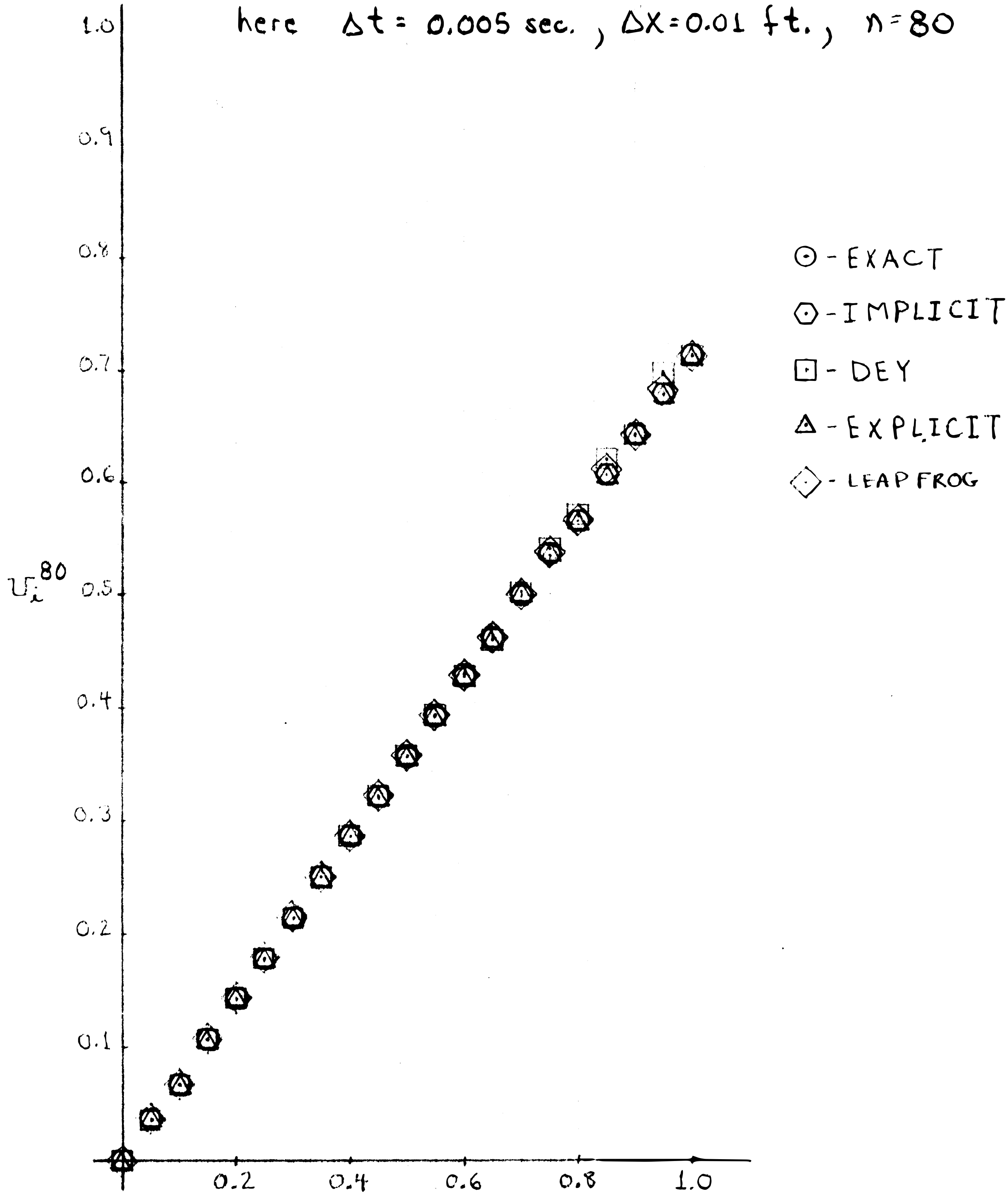


GRAPH 10

SOLUTION OF THE GAS DYNAMICS EQUATION

$$U_i^n = U(x_i, t_n) \quad , t_n = n \cdot \Delta t \quad , X_i = i \cdot \Delta X$$

here $\Delta t = 0.005$ sec., $\Delta X = 0.01$ ft., $n = 80$

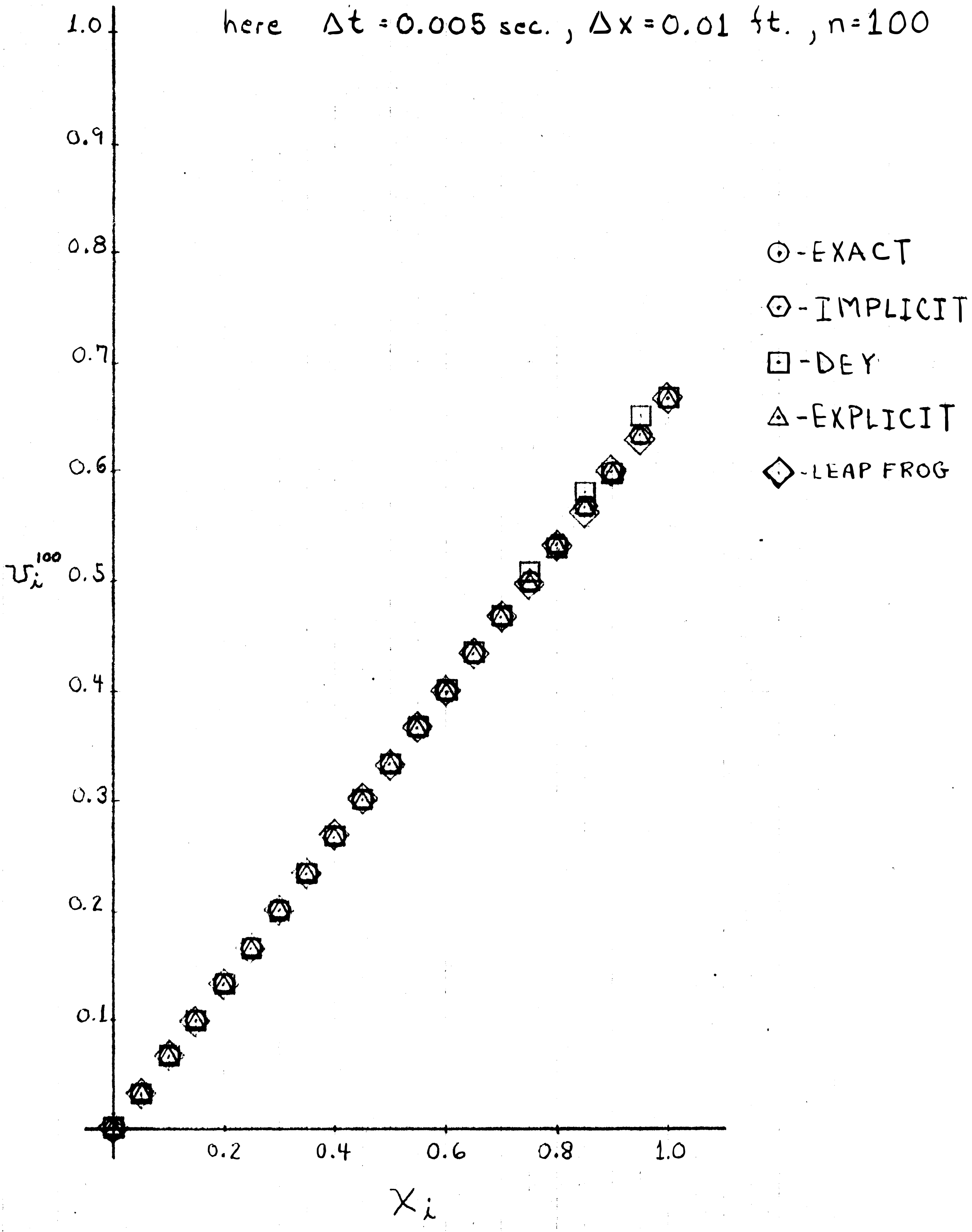


GRAPH 11

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

here $\Delta t = 0.005$ sec. , $\Delta x = 0.01$ ft. , $n = 100$

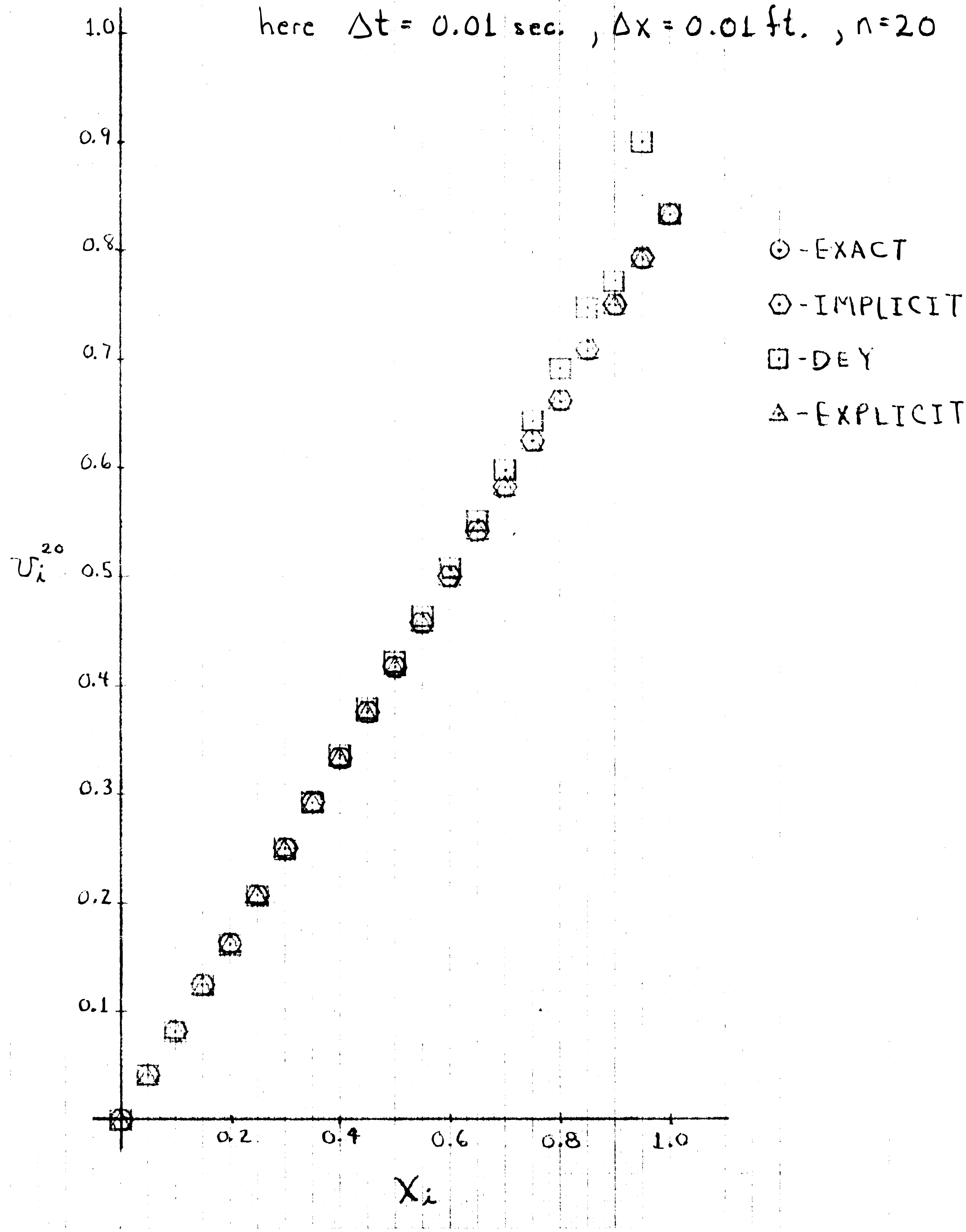


GRAPH 12

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = (x_i, t_n)$, $t_n = n \cdot \Delta t$, $X_i = i \Delta X$

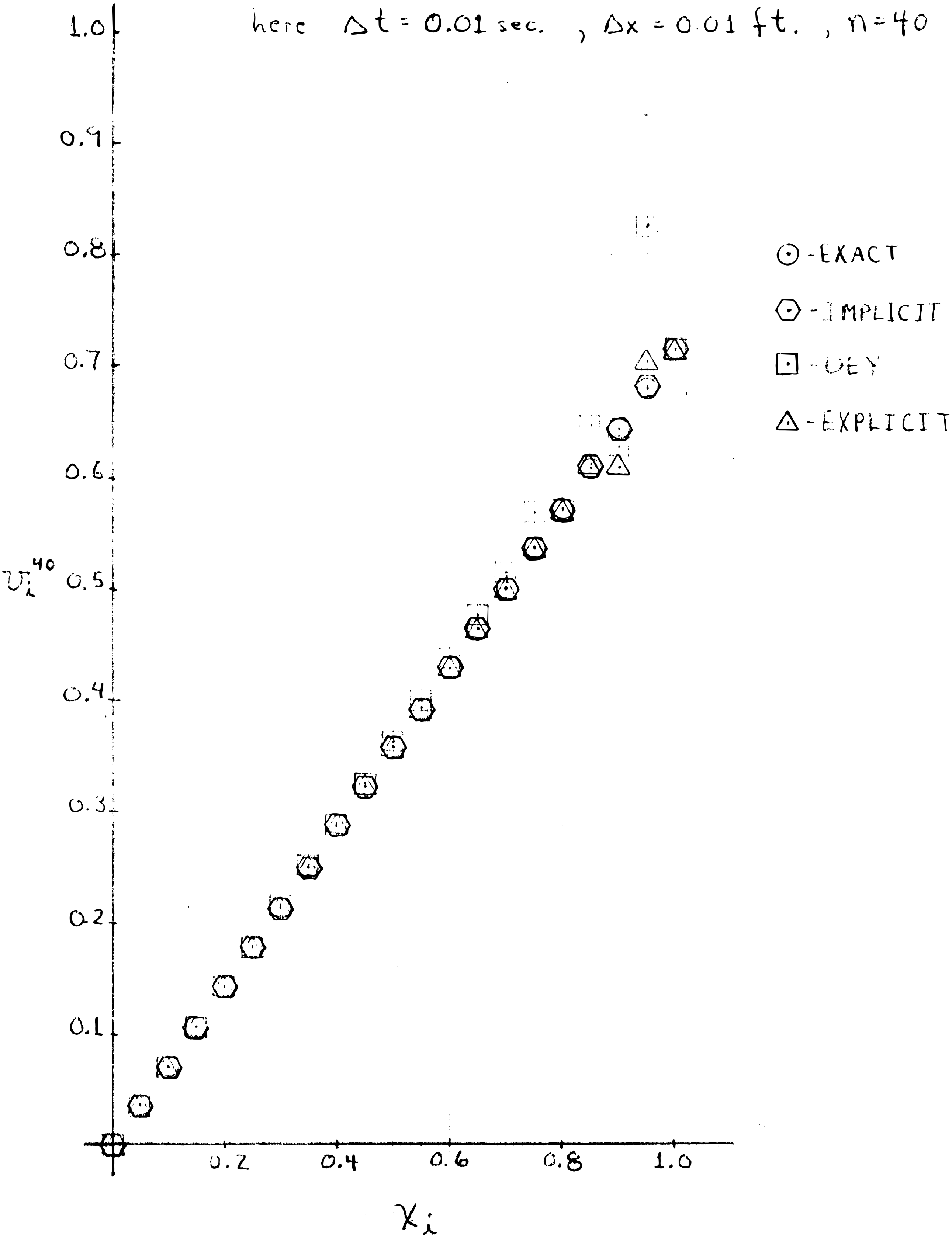
here $\Delta t = 0.01$ sec. , $\Delta x = 0.01$ ft. , $n = 20$



GRAPH 13

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

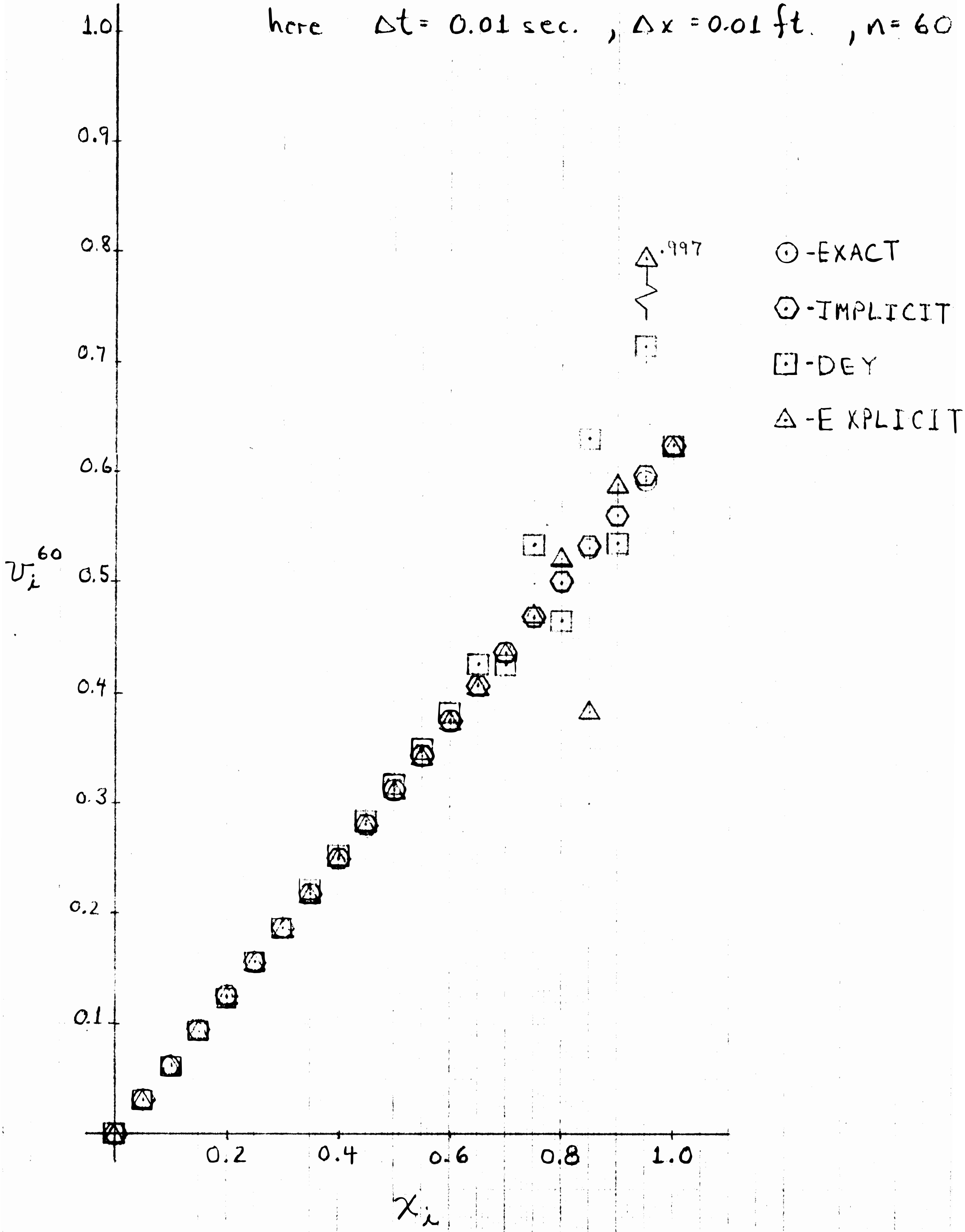


GRAPH 14

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $X_i = i \cdot \Delta x$

here $\Delta t = 0.01$ sec. , $\Delta x = 0.01$ ft. , $n = 60$

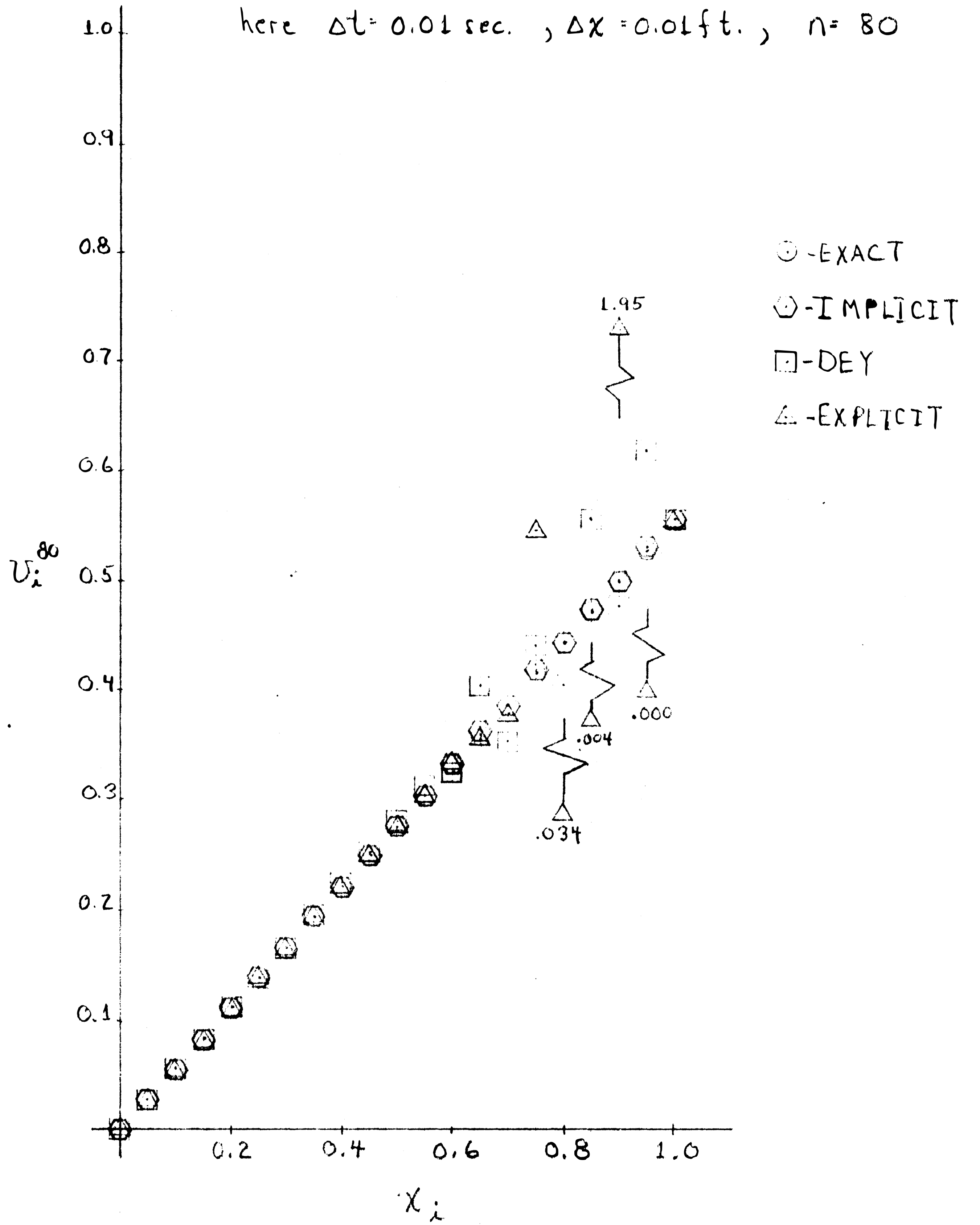


GRAPH 15

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

here $\Delta t = 0.01$ sec. , $\Delta x = 0.01$ ft. , $n = 80$

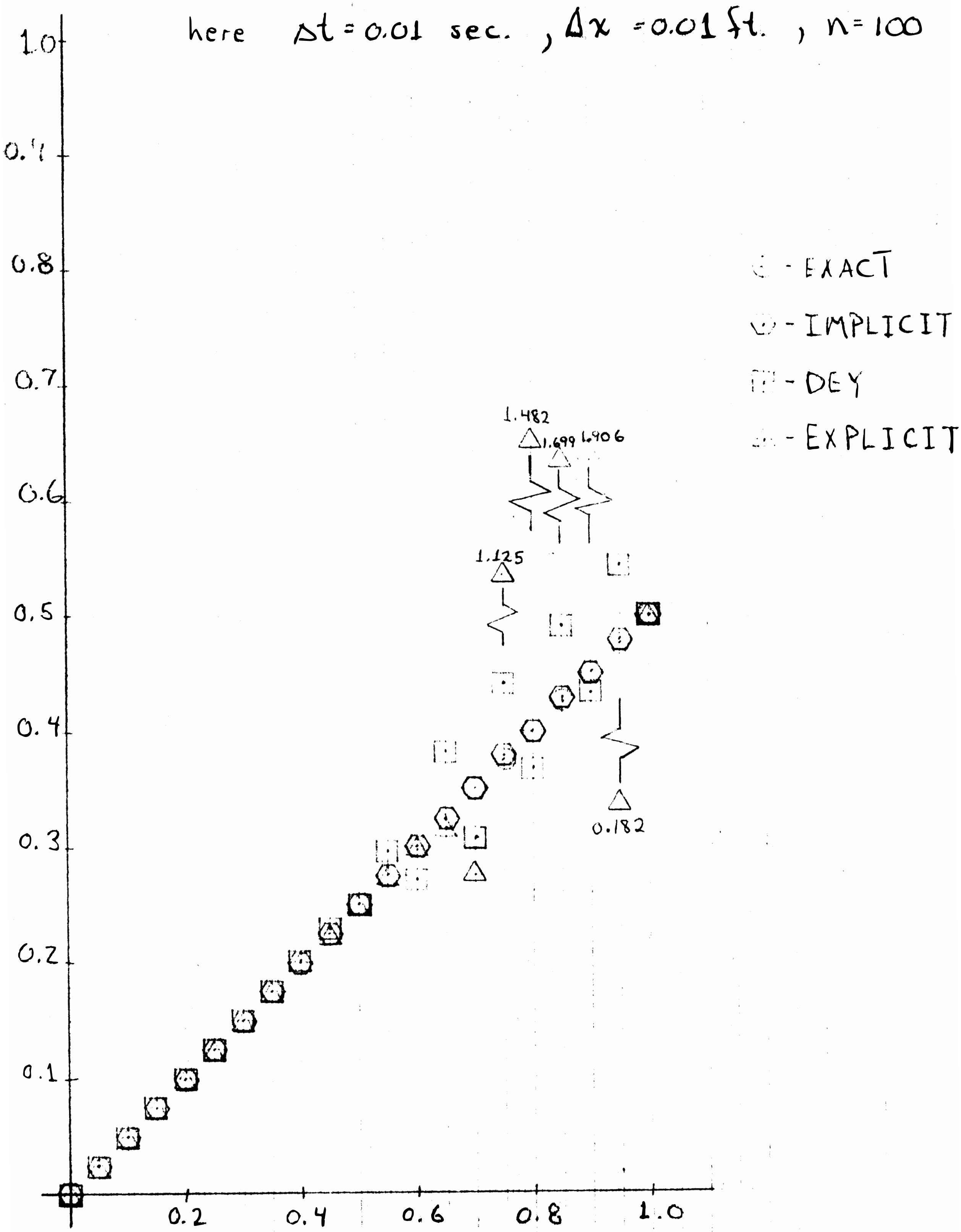


GRAPH 16

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

here $\Delta t = 0.01$ sec. , $\Delta x = 0.01$ ft. , $n = 100$

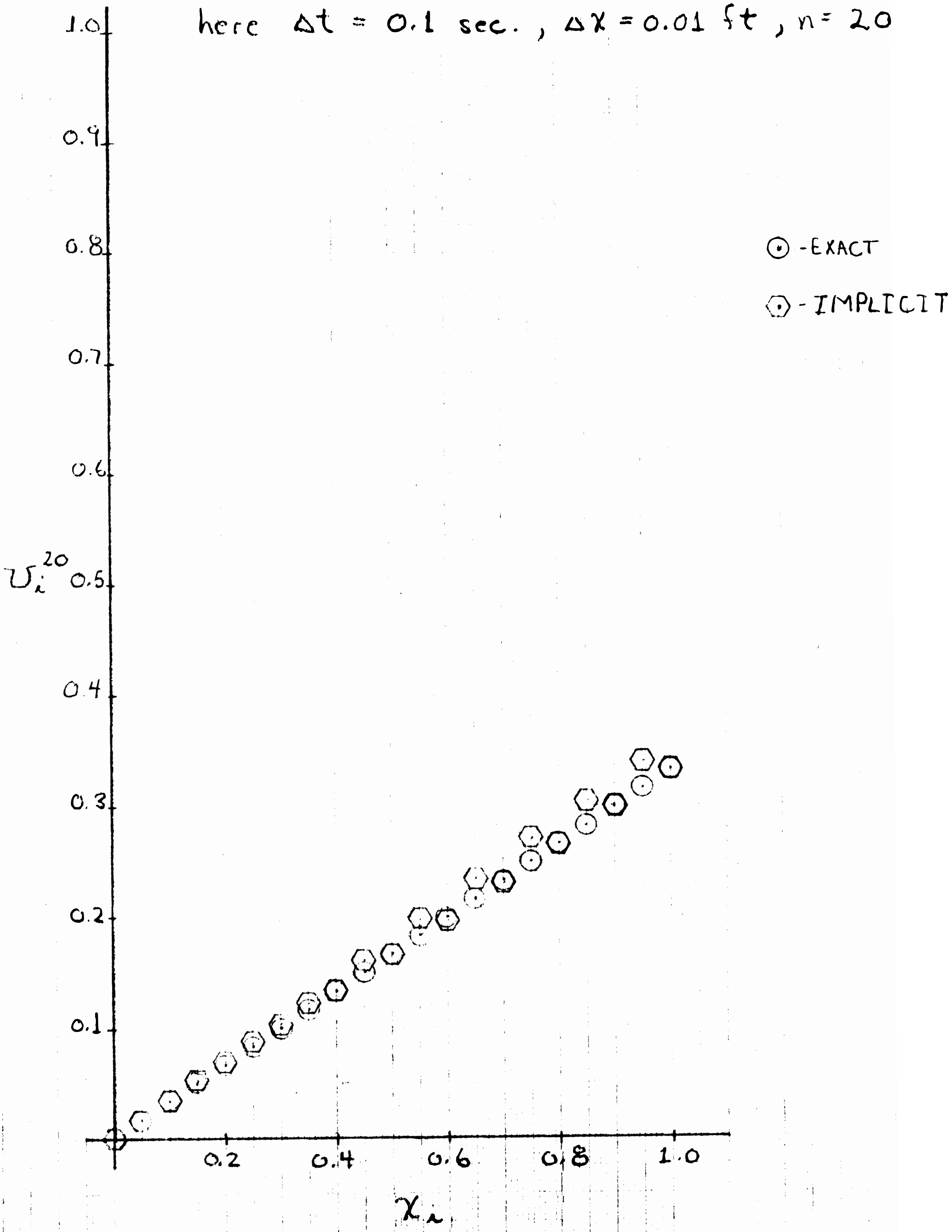


GRAPH 17

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

here $\Delta t = 0.1$ sec. , $\Delta x = 0.01$ ft , $n = 20$

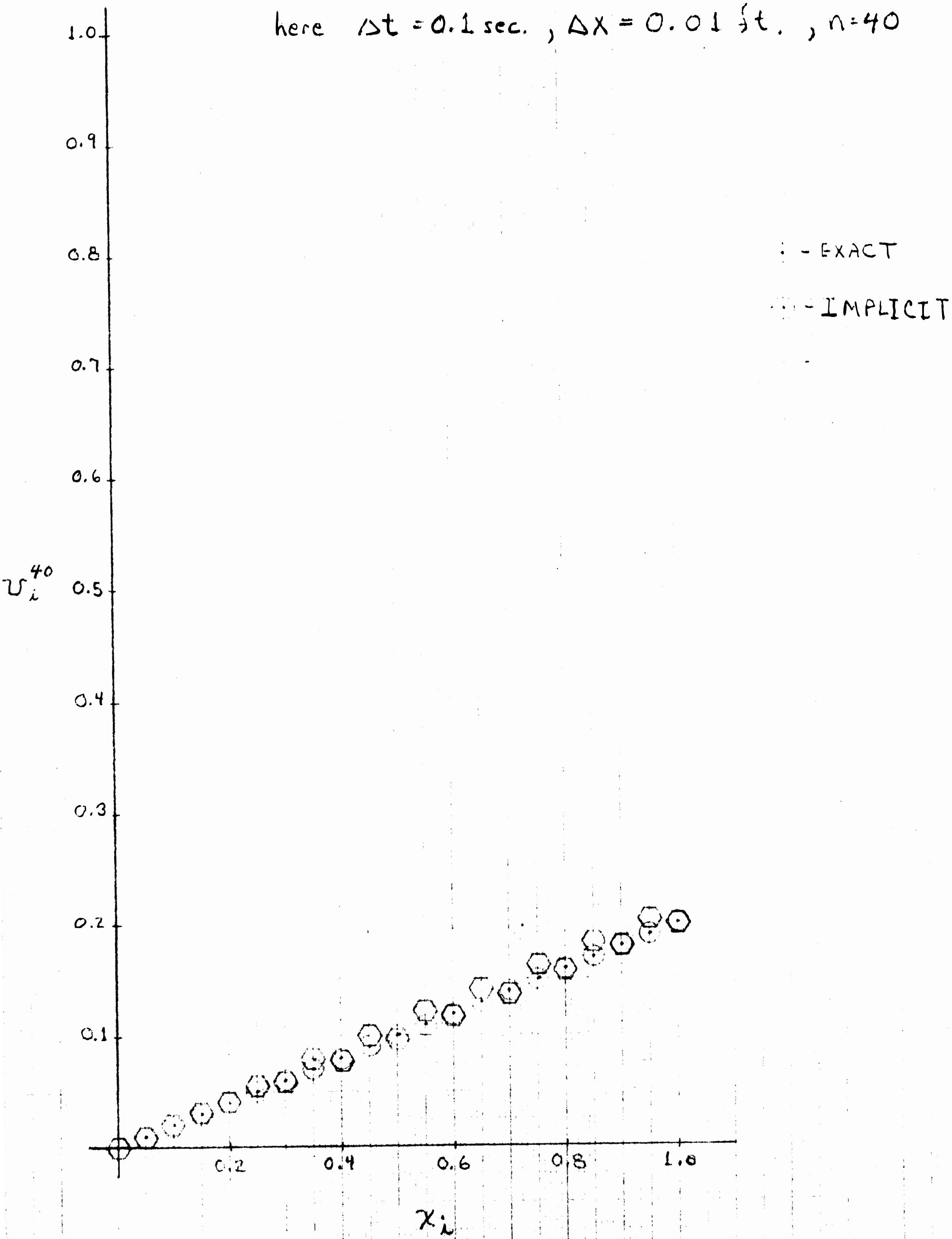


GRAPH 18

SOLUTION OF THE GAS DYNAMICS EQUATION

$v_i^n = v(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

here $\Delta t = 0.1$ sec. , $\Delta x = 0.01$ ft. , $n = 40$

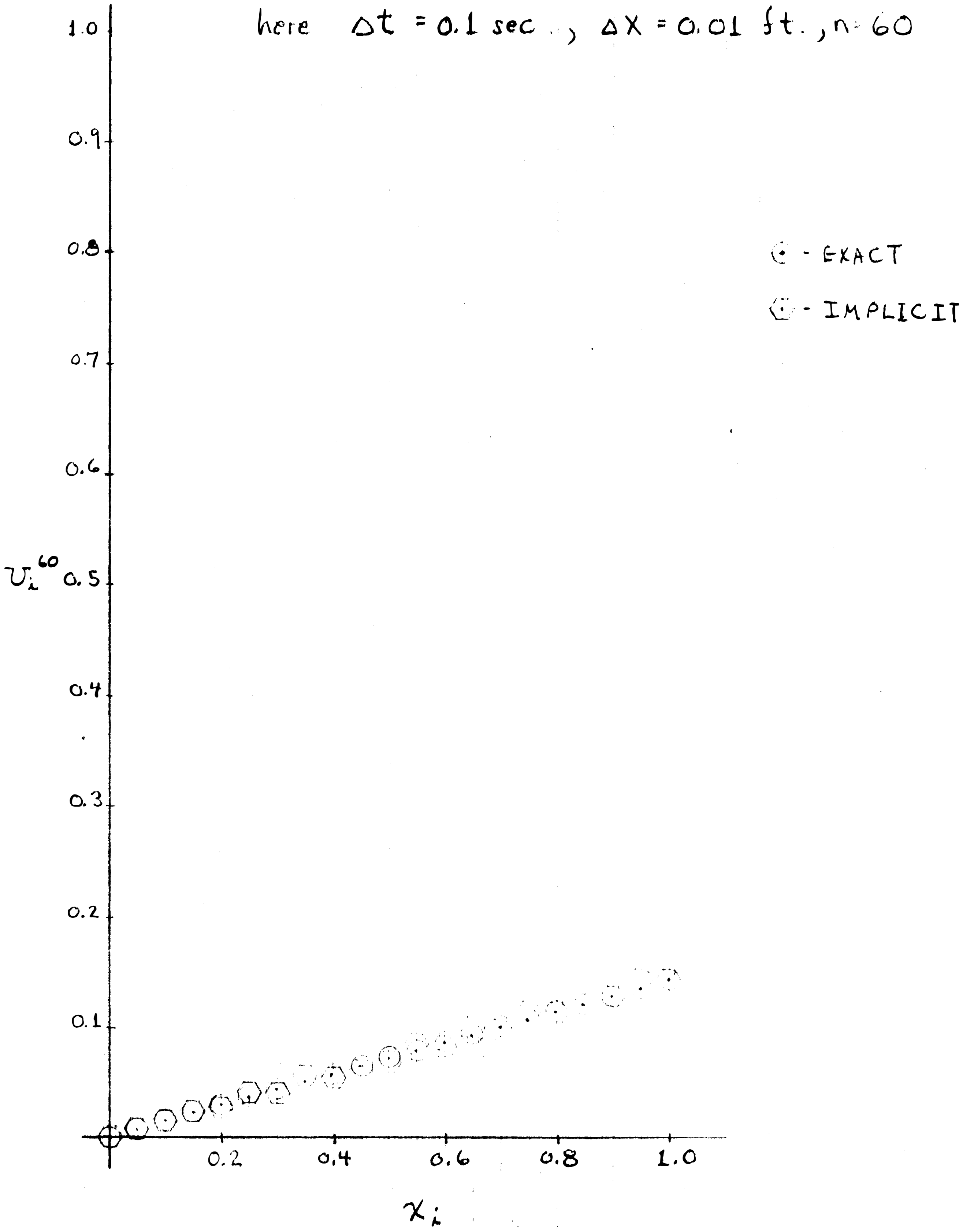


GRAPH 19

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$

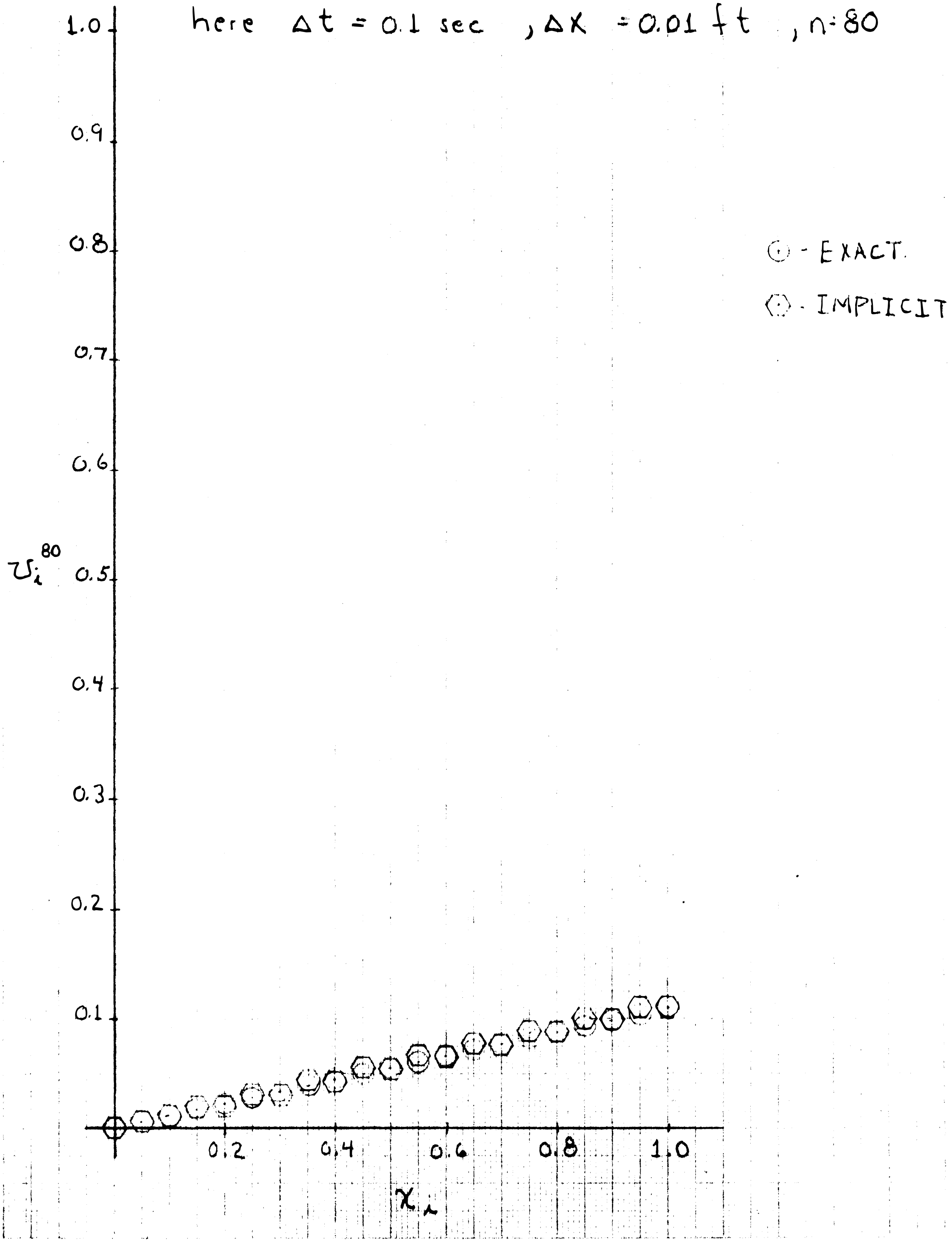
here $\Delta t = 0.1$ sec. , $\Delta X = 0.01$ ft. , $n = 60$



GRAPH 20

SOLUTION OF THE GAS DYNAMICS EQUATION

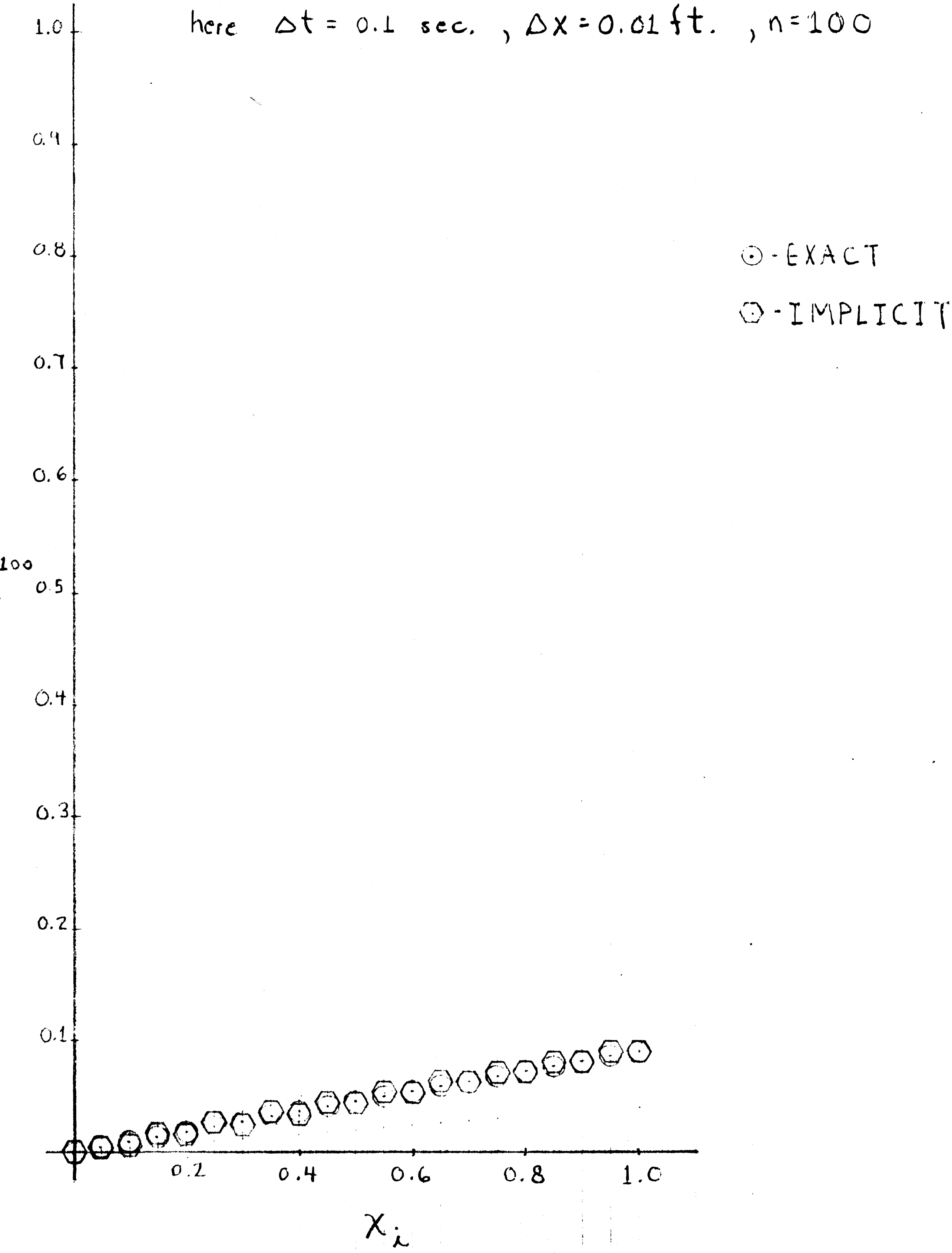
$v_i^n = v(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta X$



GRAPH 21

SOLUTION OF THE GAS DYNAMICS EQUATION

$U_i^n = U(x_i, t_n)$, $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$



REFERENCES

Ames, W. F. Numerical Methods for Partial Differential Equations (Barnes & Noble, 1969).

Briley, W. R. and McDonald, H. "Solution of Three-dimensional Compressible Navier Stokes Equations by an Implicit Technique," from Lecture Notes in Physics, June 24-28, 1974.

Dey, S.K. Numerical Studies of the Navier-Stokes Equations with an Application to a Circular Cylinder (Research Note, Mississippi State University, May 1969).

Isaacson, E. and Keller, H.B. Analysis of Numerical Methods (John Wiley and Sons, Inc., 1966).

Miller, E. L. Predictor-Corrector Studies of Burgers' Model of Turbulent Flow (University of Delaware, Master Thesis, 1966).

Mitchell, A. R. Computational Methods in Partial Differential Equations (John Wiley and Sons, Inc., 1969).

Roache, P. J. Computational Fluid Dynamics (Albuquerque, New Mexico: Hermosa Publishers, 1972).

Thompson, J. F. Numerical Solution of the Incompressible, Two-dimensional, Time Dependent Navier-Stokes Equations for a Body Oscillating in Pitch in a Moving Fluid (Research Note, Mississippi State University, October 1968).